HADAMARD PRODUCTS OF SCHLICHT FUNCTIONS

LAWRENCE ZALCMAN[1]

1. **Introduction.** Let $D=\{z \mid |z|<1\}$ be the open unit disc and let $A(D)$ be the set of all functions analytic in D. A function $f \in A(D)$ is said to have valence p in D (p a natural number or ∞) if the equation

$$f(z) = x \tag{1}$$

has, for each fixed x, no more than p distinct roots and if, for some value $x=x_0$, (1) has exactly p distinct roots. Denote by $V_D(1)$ the subset of $A(D)$ consisting of the schlicht functions (i.e., the functions of valence 1) of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{2}$$

Recently, A. W. Goodman showed that the sum or product of two functions in $V_D(1)$ may have valence ∞ [2]. In this paper, we are concerned with the behavior of $V_D(1)$ under a different composition, the familiar Hadamard product

$$\left(\sum_{n=1}^{\infty} a_n z^n, \sum_{n=1}^{\infty} b_n z^n\right) \xrightarrow{*} \sum_{n=1}^{\infty} a_n b_n z^n, \qquad a_1 = b_1 = 1. \tag{3}$$

Our principal result is that the Hadamard product of two functions in $V_D(1)$ may have valence ∞; moreover, the functions involved may be taken to satisfy certain fairly strong "regularity" requirements (details are in §2). In §3, we state some complementary results due to Robertson. Finally, in §4 we mention a related, apparently more difficult, problem.

2. **The main example.** Let $\epsilon>0$ be given. We shall construct a function

$$h(z) = z + \sum_{n=2}^{\infty} c_n z^n \tag{4}$$

such that

(i) the c_n are real,

(ii) $\sum_{n=2}^{\infty} |c_n| \leqq 1/2+\epsilon$,

Received by the editors March 4, 1967.

[1] National Science Foundation Graduate Fellow.

(iii) $h(z)$ has infinite valence.

Suppose this construction has been accomplished and that $\epsilon \leqq 1/2$. Let

$$f(z) = \frac{z}{(1-z)^2} = z + \sum_{n=2}^{\infty} nz^n$$
$$g(z) = z + \sum_{n=2}^{\infty} \frac{c_n}{n} z^n. \tag{5}$$

Clearly, f is schlicht, starlike, and typically real. Since $\sum_{n=2}^{\infty} n|c_n/n| = \sum_{n=2}^{\infty} |c_n| \leqq 1$, g is also schlicht and starlike (see, for instance, [5]); furthermore, g is typically real since the c_n are real. Finally, $(f*g)(z) = h(z)$, a function of infinite valence. It remains to construct h as required.

Let $\{x_n\}$ be a sequence of real numbers, $0 < x_n < 1$, such that $\prod_{n=1}^{\infty} x_n$ does not diverge to 0. Let

$$B(z) = \prod_{n=1}^{\infty} \frac{x_n - z}{1 - x_n z} \tag{6}$$

be the Blaschke product associated with $\{x_n\}$. B is analytic on D and satisfies $|B(z)| < 1$ there. Also,

$$B'(z) = \sum_{n=1}^{\infty} \left\{ \frac{x_n^2 - 1}{(1 - x_n z)^2} \prod_{j \neq n} \frac{x_j - z}{1 - x_j z} \right\} \tag{7}$$

so that

$$|B'(z)| \leqq 4 \sum_{n=1}^{\infty} \frac{1 - x_n^2}{|1 - z|^2}. \tag{8}$$

Given $\delta > 0$, we can choose the $\{x_n\}$ so that

$$1 - \delta < \prod_{n=1}^{\infty} x_n = B(0),$$
$$|(z-1)^2 B'(z)| \leqq 4 \sum_{n=1}^{\infty} (1 - x_n^2) < \delta, \qquad z \in D, \tag{9}$$
$$|B''(0)| < \delta.$$

Let $F(z) = (z-1)^2 B(z)$. Then

$$\begin{aligned} |F'(z)| &= |2(z-1)B(z) + (z-1)^2 B'(z)| \\ &\leqq 2|(z-1)B(z)| + |(z-1)^2 B'(z)| < 4 + \delta \end{aligned} \tag{10}$$

for $z \in D$. Thus $F'(z)$ is a bounded analytic function on D. Let

$$F'(z) = \sum_{n=0}^{\infty} f_n z^n. \tag{11}$$

We have

$$\left(\sum_{n=0}^{\infty} |f_n|^2\right)^{1/2} = \lim_{r \to 1} \left(\frac{1}{2\pi} \int_0^{2\pi} |F'(re^{i\theta})|^2 \, d\theta\right)^{1/2} < 4 + \delta. \tag{12}$$

Let

$$\begin{aligned} h(z) &= \frac{F(z) - F(0)}{F'(0)} = z + \frac{f_1}{2f_0} z^2 + \cdots \\ &= \frac{1}{f_0} \sum_{n=0}^{\infty} \frac{f_n}{n+1} z^{n+1}. \end{aligned} \tag{13}$$

Clearly $h(z)$ has infinite valence in D; indeed,

$$h(x_n) = -F(0)/F'(0), \qquad n = 1, 2, \cdots. \tag{14}$$

Now

$$f_0 = -2B(0) + B'(0), \qquad f_1 = 2B(0) - 4B'(0) + B''(0) \tag{15}$$

and the expressions for the f_n, $n \geqq 2$, involve $B'(0)$, $B''(0)$, $\cdots$ but not $B(0)$. Thus for fixed N we can arrange that

$$\sum_{n=2}^{N-1} \left|\frac{f_n}{n+1}\right| < \delta \tag{16}$$

by an appropriate choice of the $\{x_n\}$ (consistent with (9)). Choose N such that

$$\left(\sum_{n=N+1}^{\infty} \frac{1}{n^2}\right)^{1/2} < \delta. \tag{17}$$

Then

$$\sum_{n=N}^{\infty} \left|\frac{f_n}{n+1}\right| \leqq \left(\sum_{n=N}^{\infty} |f_n|^2\right)^{1/2} \left(\sum_{n=N+1}^{\infty} \frac{1}{n^2}\right)^{1/2} < (4 + \delta)\delta. \tag{18}$$

From (9) and (15) we have

$$\begin{aligned} |f_0| &> 2|B(0)| - |B'(0)| > 2 - 3\delta, \\ |f_1| &< 2|B(0)| + 4|B'(0)| + |B''(0)| < 2 + 6\delta. \end{aligned} \tag{19}$$

Thus

$$\frac{1}{|f_0|}\sum_{n=1}^{\infty}\left|\frac{f_n}{n+1}\right| = \frac{1}{|f_0|}\left\{\frac{|f_1|}{2}+\sum_{n=2}^{N-1}\frac{|f_n|}{n+1}+\sum_{n=N}^{\infty}\frac{|f_n|}{n+1}\right\} < \frac{1}{2-3\delta}\{1+8\delta+\delta^2\}. \tag{20}$$

Choosing δ so small that $(19\delta+2\delta^2)/(4-6\delta)<\epsilon$, we obtain

$$\sum_{n=2}^{\infty}|c_n| = \sum_{n=1}^{\infty}\frac{1}{|f_0|}\frac{|f_n|}{n+1} < \frac{1}{2}+\epsilon. \tag{21}$$

It is easy to see that we can choose $g(z)$ to be starlike at least of order $\alpha=(2/3)-\eta$, where $\eta>0$ can be made arbitrarily small. Indeed, by a result in [5], it is enough to show

$$\sum_{n=2}^{\infty}(n-\alpha)\frac{|c_n|}{n} \leqq 1-\alpha. \tag{22}$$

Choose $h(z)=z+\sum_{n=2}^{\infty}c_nz^n$ as above with $|c_2|>\frac{1}{2}+\epsilon/2$ and $\sum_{n=2}^{\infty}|c_n|<\frac{1}{2}+\epsilon$, where $10\,\epsilon/(9-3\epsilon)<\eta$. An easy computation then shows that (22) holds.

We can collect the results of this section into

THEOREM 1. *Let $\eta>0$. There exist functions f, $g\in V_D(1)$, starlike and typically real, such that $f*g$ has infinite valence in D. g may be taken to have an absolutely convergent Taylor series and to be starlike at least of order $(2/3)-\eta$.*

3. **Further results.** Some positive results due to Robertson [6], [7] lend perspective to the example in §2. We state them without proof.

THEOREM 2. *Let f, $g\in V_D(1)$ be typically real and convex in the direction of the imaginary axis. Then $f*g\in V_D(1)$, and $f*g$ is typically real and convex in the direction of the imaginary axis.*

THEOREM 3. *Let*

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nz^n, \qquad g(z)=\frac{1}{z}+\sum_{n=1}^{\infty}b_nz^n \tag{23}$$

be regular and schlicht in $0<|z|<1$. Then

$$h(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nb_nz^n \tag{24}$$

is regular and schlicht in $0<|z|<1$.

4. A related problem. Before closing, we should mention a related problem. Define a new composition on pairs of functions in $V_D(1)$ by

$$\left(\sum_{n=1}^{\infty} a_n z^n, \sum_{n=1}^{\infty} b_n z^n\right) \xrightarrow{\otimes} \sum_{n=1}^{\infty} \frac{a_n b_n}{n} z^n.$$

Robertson [6] showed that if f and g ($\in V_D(1)$) are typically real then so is $f \otimes g$. On the other hand, $V_D(1)$ is not closed under $\otimes$; proofs of this fact are in [1], [3], and [4]. If $f, g \in V_D(1)$, can $f \otimes g$ have infinite valence in D?

References

1. B. Epstein and I. J. Schoenberg, *On a conjecture concerning schlicht functions*, Bull. Amer. Math. Soc. **65** (1959), 273–275.

2. A. W. Goodman, *The valence of sums and products*, Canad. J. Math. (to appear).

3. W. K. Hayman, *On the coefficients of univalent functions*, Proc. Cambridge Philos. Soc. **55** (1959), 373–374.

4. C. Loewner and E. Netanyahu, *On some compositions of Hadamard type in classes of analytic functions*, Bull. Amer. Math. Soc. **65** (1959), 284–286.

5. E. P. Merkes, M. S. Robertson and W. T. Scott, *On products of starlike functions*, Proc. Amer. Math. Soc. **13** (1962), 960–964.

6. M. S. Robertson, *Applications of a lemma of Fejer to typically real functions*, Proc. Amer. Math. Soc. **1** (1950), 555–561

7. M. S. Robertson, *Convolutions of schlicht functions*, Proc. Amer. Math. Soc. **13** (1962), 585–589.

Massachusetts Institute of Technology