ON A CLASS OF PERTURBATIONS OF THE HARMONIC OSCILLATOR[1]

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1. The following theorem, concerning solutions of

$$y'' + [1 + f(x) + h(x) \cos 2\eta x]y = 0, \tag{1.1}$$

was proved by Atkinson in the cases $\alpha = 1$ (where the sum in (1.6) is empty) and $\alpha = 2$; see [1, p. 349 and p. 355]. In [3], Kelman and Madsen formulated the general result ($\alpha = 1, 2, \cdots$) and proved it using different methods.

THEOREM 1.1 [3]. *Let $f(x) \in L^1[0, \infty)$; $h(x)$ of bounded variation on $[0, \infty)$ for which there exists an integer $\alpha > 0$ satisfying*

$$\int^{\infty} |h|^{\alpha} dx = \infty \quad \text{and} \quad \int^{\infty} |h|^{\alpha+1} dx < \infty; \tag{1.2}$$

$\eta > 0$ a constant satisfying

$$0 < \eta \neq k/j, \quad \text{where} \quad 1 \leqq k \leqq \alpha - 1 \quad \text{and} \quad 1 \leqq j \leqq \alpha, \tag{1.3}$$

and, if α is odd,

$$0 < \eta \neq \alpha/j \qquad \text{for } j = 1, 3, \cdots, \alpha. \tag{1.4}$$

Then for even integers $2j$, $2 \leqq 2j \leqq \alpha$, there are real-valued rational functions $c_{2j} = c_{2j}(\eta)$ of η finite on (1.3)–(1.4), with the following property: There exists a one-to-one correspondence between solutions $y(x)$ of (1.1) and pairs of constants (a_1, a_2) such that

$$\begin{aligned} y &= a_1 \sin \theta(x) + a_2 \cos \theta(x) + o(1), \\ y' &= a_1 \cos \theta(x) - a_2 \sin \theta(x) + o(1), \end{aligned} \tag{1.5}$$

$$\theta(x) = x + \sum_{2 \leqq 2j \leqq \alpha} c_{2j} \int_0^x h^{2j}(s) ds. \tag{1.6}$$

For related (less precise) results, see references in [3] to J. G. van der Corput.

Using a device from Hartman [2], we shall give a somewhat more

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transparent proof and, at the same time, replace (1.1) by the more general equation

$$(1.7)\qquad y'' + \left[1 + 2f(x) + 2\sum_{m=0}^{M} h_m(x)\cos(2\eta_m x + \gamma_m)\right] y = 0.$$

It should be pointed out that Atkinson [1] had used a related method for obtaining Theorem 1.1 for $\alpha=2$ and had noted that this argument can be used to show the validity of the following result for $\alpha=2$, $h_0\equiv 0$, $\gamma_1=\gamma_2=\cdots=\gamma_M=0$.

THEOREM 1.2. *Let* $f(x)\in L^1[0,\infty)$; $h_0(x),\cdots,h_M(x)$ *functions of bounded variation on* $[0,\infty)$ *for which there is an integer* $\alpha>0$ *satisfying*

$$(1.8)\qquad \sum_{m=0}^{M}\int^{\infty} |h_M|^{\alpha+1}dx < \infty;$$

let $\eta_0=0<\eta_1\leqq\cdots\leqq\eta_m$ *be constants with the property that*

$$(1.9)\qquad |\eta_{m(1)} \pm \eta_{m(2)} \pm \cdots \pm \eta_{m(\nu)}| \neq \tau, \qquad \text{where } \tau = 1,\cdots,\nu,$$

whenever

$$0 \leqq m(j) \leqq M, \qquad 1\leqq \nu\leqq\alpha, \qquad \int^{\infty}\prod_{j=1}^{\nu}|h_{m(j)}|\,dx = \infty;$$

finally, $\gamma_0=0$ *and* $\gamma_1,\cdots,\gamma_M$ *are arbitrary constants. Then there exists a one-to-one correspondence between solutions* $y(x)$ *of* (1.7) *and pairs of constants* (a_1, a_2) *such that*

$$(1.10)\qquad \begin{aligned} y &= a_1\sin\theta(x) + a_2\cos\theta(x) + o(1),\\ y' &= a_1\cos\theta(x) - a_2\sin\theta(x) + o(1),\end{aligned}$$

$$(1.11)\qquad \begin{aligned}\theta(x) &= x + \int_0^x h_0 ds\\ &\quad + \sum_{\mu=2}^{\alpha}\sum_{I[\mu]} c_{I[\mu]}(\cos\Gamma_{I[\mu]})\int_0^x\prod_{j=1}^{\mu} h_{m(j)}ds,\\ I[\mu] &= (m(1), \pm m(2),\cdots,\pm m(\mu)),\\ \Gamma_{I[\mu]} &= \gamma_{m(1)} \pm \gamma_{m(2)} \pm \cdots \pm \gamma_{m(\mu)},\end{aligned}$$

$c_{I[\mu]} = c_{m(1),\pm m(2),\cdots,\pm m(\mu)}$ *are rational functions of* $(\eta_{m(1)},\cdots,\eta_{m(\mu)})$ *which are finite for* (1.9), *and* $\sum_{I[\mu]}$ *is the sum over the set of indices* $I[\mu] = (m(1),\pm m(2),\cdots,\pm m(\mu))$ *for which* $0\leqq m(j)\leqq M$,

$$\eta_{m(1)} \pm \eta_{m(2)} \pm \cdots \pm \eta_{m(\mu)} = 0 \text{ and}$$

(1.12) $$\int_0^\infty \prod_{j=1}^{\mu} h_{m(j)} ds \text{ is not convergent.}$$

REMARK 1. The rational functions $c_{m(1),\pm m(2),\cdots,\pm m(\mu)}$ are independent of the solution $y(x)$, of the function $f(x)$, and of the functions $(h_0, \cdots, h_M)$ within the class of sets of functions $(h_0, \cdots, h_M)$ for which the convergence properties of the integrals $\int^\infty \prod h_{m(j)} dx$ do not vary. Note that if $\mu = 1$, then (1.12) can hold only for $m(1) = 0$.

REMARK 2. If, for some k on $0 \leqq k \leqq M$, $h_k(x) \equiv 0$ or more generally, $\int^\infty |h_k| dx < \infty$, then the corresponding term $2h_k(x) \cos(2\eta_k x + \gamma_k)$ in (1.7) can be considered part of the term $2f(x)$. In this case, no $m(j) = k$ occurs in (1.9), (1.11), and (1.12).

REMARK 3. In the special case (1.1) of (1.7), we have $h_0(x) \equiv 0$, $\eta_{m(j)} = \eta$ for all $j \geqq 1$, and (1.9) is equivalent to (1.3)–(1.4). Also the first part of (1.12) cannot hold unless $\mu = 2j$ is even (and there are j signs $+$ and j signs $-$), so that (1.10)–(1.11) reduce to (1.5)–(1.6). (In order to see the equivalence of (1.9) and (1.3)–(1.4), let μ be the number of $+$ signs in $|+\eta_{m(1)} \pm \eta_{m(2)} \pm \cdots \pm \eta_{m(\nu)}|$, so that the conditions on η become $0 < |2\mu - \nu| \eta \neq \tau$ for $\mu, \tau = 1, \cdots, \nu$ and $\nu = 1, \cdots, \alpha$. If α is odd, this reduces to (1.3)–(1.4). If α is even, this reduces to (1.3)–(1.4) and the apparently additional conditions $0 < \eta \neq \alpha/j$ for $j = 2, 4, \cdots, \alpha$. But these additional conditions are contained in (1.3).

2. **Proof of Theorem 1.2.** Introduce the abbreviation

$$F(x) = \sum_{m=0}^{M} h_m(x) \cos(2\eta_m x + \gamma_m).$$

From the Prüfer transformation

(2.1) $$y(x) = r(x) \sin \phi(x), \qquad y'(x) = r(x) \cos \phi(x),$$

and (1.7), we get

(2.2) $$d \log r = -F(x) \sin 2\phi dx - f(x) \sin 2\phi dx,$$

(2.3) $$d\phi = dx + F(x)(1 - \cos 2\phi) dx + f(x)(1 - \cos 2\phi) dx.$$

Following a device of Hartman [2], the last relation will also be used in the form

(2.4) $$ds = d\phi(s) - f(s)(1 - \cos 2\phi(s)) ds - F(s)(1 - \cos 2\phi(s)) ds.$$

In view of (2.2),

$$\text{(2.5)} \quad \log r(x) = c + o(1) + \sum_{m=0}^{M} \int_0^x h_m \cos(2\eta_m s + \gamma_m) \sin 2\phi(s) ds;$$

also, we have

$$\phi(x) = x + c + o(1) + \int_0^x F ds - \int_0^x F \cos 2\phi ds.$$

Since h_m is of bounded variation on $[0, \infty)$,

$$\text{(2.6)} \quad \int_0^\infty h_m(x) \cos(2\eta_m x + \gamma_m) dx = \lim_{T\to\infty} \int_0^T \qquad \text{exists if } \eta_m \neq 0.$$

Thus

$$\text{(2.7)} \quad \begin{aligned} \phi(x) = x + c + o(1) &+ \int_0^x h_0 ds \\ &- \sum_{m=0}^{M} \int_0^x h_m \cos(2\eta_m s + \gamma_m) \cos 2\phi ds. \end{aligned}$$

In (2.5), (2.7) and below, c will always denote a constant not necessarily the same one. The analogue of (2.6) will be used repeatedly below.

LEMMA 2.1. *Let $\phi(x)$ be as above; $g(x)$ a function of bounded variation on $[0, \infty)$, $g(x) = o(1)$ as $x \to \infty$; $\sigma, \tau, \gamma^0, \gamma$ and δ real constants such that*

$$\text{(2.8)} \quad |\sigma| \neq |\tau|, \qquad \tau \neq 0.$$

Then, as $x \to \infty$,

$$\text{(2.9)} \quad \begin{aligned} &\int_0^x g(s) \cos(2\sigma s + \gamma^0 - \gamma) \cos(2\tau\phi - \delta) ds \\ &\qquad = c + o(1) + 4(\tau^2 - \sigma^2)^{-1}\{\cdots\}, \end{aligned}$$

where $\{\cdots\}$ is the expression

$$\text{(2.10)} \quad \begin{aligned} \{\cdots\} = (\sigma\tau) &\sum_{m=0}^{M} \sum_{j=0}^{1} \sum_{k=-1}^{1} \epsilon_k \int_0^x g h_m \\ &\times \sin[2(\sigma + (-1)^j \eta_m)s + \gamma^0 - \gamma + (-1)^j \gamma_m] \\ &\times \sin[2(\tau + k)\phi - \delta] ds + \tau^2 \sum_{m=0}^{M} \sum_{j=0}^{1} \sum_{k=-1}^{1} \epsilon_k \int_0^x g h_m \\ &\times \cos[2(\sigma + (-1)^j \eta_m)s + \gamma^0 - \gamma + (-1)^j \gamma_m] \\ &\times \cos[2(\tau + k)\phi - \delta] ds, \end{aligned}$$

$\epsilon_{\pm 1}=1$ *and* $\epsilon_0=-2$.

We shall only need the cases $\gamma=\delta=0$ and $\gamma=\delta=\pi/2$ for the asymptotic behavior of $\phi(x)$, and the cases $\gamma=\pi/2$, $\delta=0$ and $\gamma=0$, $\delta=\pi/2$ for $r(x)$.

PROOF. Let I denote the integral on the left of (2.9). Replace ds in I by its value in (2.4) and integrate the resulting first term by parts to obtain

$$
\begin{aligned}
I &= c + o(1) + (\sigma/\tau)\int_0^x g(s)\sin(2\sigma s+\gamma^0-\gamma)\sin(2\tau\phi-\delta)ds \\
&\quad - \int_0^x g(s)\cos(2\sigma s+\gamma^0-\gamma)\cos(2\tau\phi-\delta)F(s)(1-\cos 2\phi)ds.
\end{aligned}
\tag{2.11}
$$

Use (2.4) in the first integral on the right of (2.11) and integrate the first term by parts,

$$
\begin{aligned}
I &= c + o(1) + (\sigma/\tau)^2 I \\
&\quad - (\sigma/\tau)\int_0^x g(s)\sin(2\sigma s+\gamma^0-\gamma)\sin(2\tau\phi-\delta)F(s)(1-\cos 2\phi)ds \\
&\quad - \int_0^x g(s)\cos(2\sigma s+\gamma^0-\gamma)\cos(2\tau\phi-\delta)F(s)(1-\cos 2\phi)ds.
\end{aligned}
$$

In view of the relations, for $\chi=\sin$ or $\chi=\cos$,

$$
\begin{aligned}
&2\chi(2\tau\phi-\delta)(1-\cos 2\phi) \\
&\qquad = -\sum_{k=-1}^{1}\epsilon_k\chi[2(\tau+k)\phi-\delta],
\end{aligned}
$$

and

$$
\begin{aligned}
&2\chi(2\sigma s+\gamma^0-\gamma)F(s) \\
&\qquad = \sum_{m=0}^{M}\sum_{j=0}^{1} h_m(s)\chi[2(\sigma+(-1)^j\eta_m)s+\gamma^0-\gamma+(-1)^j\gamma_m],
\end{aligned}
$$

formula (2.9) follows. This completes the proof.

On $\phi(x)$. We now show, by an induction on ν for $1\leqq\nu\leqq\alpha+1$, that, under the assumption (1.9), $\phi(x)$ has the form

$$
\begin{aligned}
\phi(x) = x + c + o(1) + \int_0^x h_0 ds \\
+ \sum_{\mu=1}^{\nu-1} \sum_{I[\mu]} c_{I[\mu]} (\cos \Gamma_{I[\mu]}) \int_0^x \prod_{j=1}^{\mu} h_{m(j)} ds \\
+ \sum_{\tau=1}^{\nu} \sum_{I[\mu]}{}' a_{\tau,I[\nu]} \int_0^x \prod_{j=1}^{\nu} h_{m(j)} \\
\times \cos(2N_{I[\nu]} s + \Gamma_{I[\nu]}) \cos 2\tau\phi ds \\
+ \sum_{\tau=1}^{\nu} \sum_{I[\nu]}{}' b_{\tau,I[\nu]} \int_0^x \prod_{j=1}^{\nu} h_{m(j)} \\
\times \sin(2N_{I[\nu]} s + \Gamma_{I[\nu]}) \sin 2\tau\phi ds,
\end{aligned}
\tag{2.12}
$$

where

$$N_{I[\nu]} = \eta_{m(1)} \pm \eta_{m(2)} \pm \cdots \pm \eta_{m(\nu)},$$

$c_{I[\mu]} = c_{m(1),\pm m[2],\cdots,\pm m(\mu)}$ and $a_{\tau,I[\nu]} = a_{\tau,m(1),\pm m(2),\cdots,\pm m[\nu]}$, $b_{\tau,I[\nu]} = b_{\tau,m(1),\pm m(2),\cdots,\pm m(\nu)}$ are rational functions of $(\eta_{m(1)}, \cdots, \eta_{m(\mu)})$ and of $(\eta_{m(1)}, \cdots, \eta_{m(\nu)})$, respectively, finite for (1.9); $\sum_{I[\mu]}$ is the sum over the sets of indices $(m(1), \pm m(2), \cdots, \pm m(\mu))$, $0 \leqq m(j) \leqq M$, for which

$$\eta_{m(1)} \pm \cdots \pm \eta_{m(\mu)} = 0 \text{ and } \int^{\infty} \prod_{j=1}^{\mu} h_{m(j)} dx \text{ is not convergent;} \tag{2.13}$$

finally $\sum'_{I[\nu]}$ is the sum over all sets $I[\nu] = (m(1), \pm m(2), \cdots, \pm m(\nu))$ for which

$$\int^{\infty} \prod_{j=1}^{\nu} | h_{m(j)} | dx = \infty. \tag{2.14}$$

The formula (2.7) can be written in the form (2.12) for $\nu = 1$. We assume (2.12) for some given ν, $1 \leqq \nu \leqq \alpha$. Then the assumption (1.9) makes Lemma 2.1 applicable to each term in the last two sums of (2.12), with $\gamma = \delta = \pi/2$ in the last sum and $\gamma = \delta = 0$ in the next to last sum, $\sigma = \eta_{m(1)} \pm \cdots \pm \eta_{m(\nu)}$, $\gamma^0 = \gamma_{m(1)} \pm \cdots \pm \gamma_{m(\nu)}$, and $g(x) = \prod h_{m(j)}$ for $j = 1, \cdots, \nu$. This shows the validity of (2.12) when ν is replaced by $\nu + 1$, since

$$\int_0^x \prod_{j=1}^{\nu+1} h_{m(j)} \cos(2N_{I[\nu+1]} s + \Gamma_{I[\nu+1]}) ds = c + o(1)$$

if the first part of (2.13) does not hold. Hence (2.12) is valid for $\nu=\alpha+1$. Thus $\phi(x)$ can be written in the form

$$\phi(x) = c^0 + o(1) + \theta(x), \tag{2.15}$$

where $\theta(x)$ is independent of the solution $y(x)$ and is given by (1.11).

On $r(x)$. Starting with (2.5), the cases $\gamma=\pi/2$, $\delta=0$ and $\gamma=0$, $\delta=\pi/2$ of Lemma 2.1 imply, by an induction on ν, $1\leqq\nu\leqq\alpha+1$, that $\log r(x)$ can be written in the form

$$\begin{aligned} \log r(x) &= c + o(1) \\ &+ \sum_{\mu=1}^{\nu-1} \sum_{I[\mu]}{}^{*} c_{I[\mu]}(\sin \Gamma_{I[\mu]}) \int_0^x \prod_{j=1}^{\mu} h_{m(j)}ds \\ &+ \sum_{\tau=1}^{\nu} \sum_{I[\nu]}{}'^{*} a_{\tau,I[\nu]} \\ &\times \int_0^x \prod_{j=1}^{\nu} h_{m(j)} \cos(2N_{I[\nu]}s + \Gamma_{I[\nu]})\sin 2\tau\phi ds \\ &+ \sum_{\tau=1}^{\nu} \sum_{I[\nu]}{}'^{*} b_{\tau,I[\nu]} \\ &\times \int_0^x \prod_{j=1}^{\nu} h_{m(j)} \sin(2N_{I[\nu]}s + \Gamma_{I[\nu]})\cos 2\tau\phi ds, \end{aligned} \tag{2.16}$$

in notation analogous to (2.12).

Thus, the case $\nu=\alpha+1$ shows that

$$r(x) = [c^1 + o(1)] \exp \rho(x), \tag{2.17}$$

where $\rho(x)$ is independent of the solution $y(x)$ and

$$\rho(x) = \sum_{\mu=1}^{\alpha} \sum_{I[\mu]}{}^{*} c_{I[\mu]}(\sin \Gamma_{I[\mu]}) \int_0^x \prod_{j=1}^{\mu} h_{m(j)}ds. \tag{2.18}$$

Completion of the proof. Let $y_1(x)$, $y_2(x)$ be two solutions of (1.7) with the Wronskian

$$y_1y_2' - y_1'y_2 \equiv 1. \tag{2.19}$$

Then, by (2.1), (2.15) and (2.17), for $j=1, 2$,

$$\begin{aligned} y_j &= [c_j^1 + o(1)]e^{\rho(x)} \sin [c_j^0 + o(1) + \theta(x)], \\ y_j' &= [c_j^1 + o(1)]e^{\rho(x)} \cos [c_j^0 + o(1) + \theta(x)], \end{aligned} \tag{2.20}$$

where (c_j^0, c_j^1) are the constants (c^0, c^1) belonging to $y_j(x)$. By (2.19),

$$[\overset{1}{c}_1\overset{1}{c}_2 + o(1)]e^{2\rho(x)} \sin(\overset{0}{c}_1 - \overset{0}{c}_2 + o(1)) \equiv 1.$$

Thus, according as

$$\overset{1}{c}_1\overset{1}{c}_2 \sin(\overset{0}{c}_1 - \overset{0}{c}_2) = 0 \quad \text{or} \quad \neq 0, \tag{2.21}$$

it follows that

$$\lim_{x\to\infty} \rho(x) = +\infty \quad \text{or} \quad \text{exists (finite).} \tag{2.22}$$

Actually, the first alternative in (2.22) cannot hold. In order to see this, consider the differential equation obtained by changing the signs of the γ_m in (1.7),

$$y'' + \left[1 + 2f(x) + 2\sum_{m=0}^{M} h_m(x)\cos(2\eta_m x - \gamma_m)\right]y = 0,$$

and let $\theta_1(x)$, $\rho_1(x)$ belong to this equation as $\theta(x)$, $\rho(x)$ in (1.11), (2.18) belong to (1.7). Then, the deduction of (2.22) shows that

$$\lim_{x\to\infty} \rho_1(x) = +\infty \quad \text{or} \quad \text{exists (finite).}$$

But $\theta(x) \equiv \theta_1(x)$ and $\rho_1(x) \equiv -\rho(x)$; thus $\rho(\infty) \neq +\infty$. Consequently, changing c^1, (2.17) becomes $r(x) = c^1 + o(1)$, $\rho(x) \equiv 0$.

Correspondingly, by the formulae following (2.19),

$$y_j = [\overset{1}{c}_j + o(1)]\sin(\overset{0}{c}_j + o(1) + \theta(x)),$$

$$y_j' = [\overset{1}{c}_j + o(1)]\cos(\overset{0}{c}_j + o(1) + \theta(x))$$

and $c_1^1 c_2^1 \sin(c_1^0 - c_2^0) \neq 0$. Thus, for a solution $y(x) \not\equiv 0$, $c^1 > 0$ in $r(x) = c^1 + o(1)$, and linearly independent solutions y_1, y_2 belong to pairs (c_1^0, c_1^1), (c_2^0, c_2^1) with $c_1^0 \neq c_2^0 \pmod{\pi}$. This completes the proof of Theorem 1.2.

References

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