A CLASS OF CURVES ON WHICH POLYNOMIALS APPROXIMATE EFFICIENTLY[1]

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A *normal curve* is defined to be a continuous curve of finite length contained in the unit square $0 \leqq x,\ y \leqq 1$. It is well known that the uniform closure of the polynomials on a normal curve, Q, is the class of all functions continuous on Q. In this note we concern ourselves with the accuracy of the approximation of continuous functions on Q by nth degree polynomials. In particular, we seek a class of normal curves on which the accuracy of the approximation by nth degree polynomials compares favorably with that of any other n^2-dimensional set of approximating functions. In order to state our result precisely, we make the following definitions:

Let Q be a compact set in E^k (Euclidean k-space). The *massivity*, $m_n(Q)$, is a sequence defined as follows: Let X_n be a set of $n+1$ elements of Q; then

$$m_n(Q) = \max_{X_n \subset Q} \min_{x_i, x_j \in X_n;\, i \neq j} |x_i - x_j|.$$

Given a function $f(x)$ defined on Q and a $\delta > 0$, the *modulus of continuity* of the function $f(x)$ is defined as usual:

$$\omega_f(\delta) = \sup_{|x-y| \leq \delta} |f(x) - f(y)|.$$

If $\omega_f(\delta) \leqq \delta$ for all $\delta > 0$, $f(x)$ is said to be a *contraction* on Q. Let $C(Q)$ be the space of all real valued continuous functions on Q and let P be a finite dimensional subspace of $C(Q)$. Let K denote the class of contractions on Q. We define the "degree of approximation" as

$$\rho(P) = \max_{f \in K} \min_{p \in P} \max_{x \in Q} |f - p|.$$

An nth degree polynomial in more than one variable is defined to be a polynomial in which the maximum degree in any single variable is less than n. Unless otherwise specified, P will denote the subspace consisting of nth degree polynomials. In this case $\rho(P)$ may be denoted ρ_n.

Jackson's classical theorem established that if Q is the interval $0 \leqq x \leqq 1$, then there exist positive constants c_1, c_2 such that

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$$c_1/n \leqq \rho_n \leqq c_2/n \qquad [2, \text{p. } 14].$$

It is also known that if Q is a normal curve then $\rho_n = O(1/n)$ [4, Lemma 5], while for P any n^2-dimensional subspace of $C(Q)$,

$$(1) \qquad \rho(P) \geqq m_{n^2}/2 \geqq c/n^2 \qquad [3], [4, \text{Lemma } 1].$$

Thus, if Q is a curve for which the degree of approximation is equal to $O(1/n^2)$, polynomials can be termed *efficient* on Q. We now state

THEOREM 1. *Let Q be the curve $y = \sum_{k=1}^{\infty} a_k x^k$, $0 \leqq x \leqq 1$, where $|a_1| \leqq 1$ and $|a_{n+1}| < |a_n|^{n^2}$. Then there exist positive constants c_1, c_2 such that*

$$c_1/n^2 \leqq \rho_n \leqq c_2/n^2.$$

In particular, polynomials are efficient on the curve given by

$$y = \sum_{n=1}^{\infty} \frac{x^n}{n^{n^{n \log n}}}, \qquad 0 \leqq x \leqq 1.$$

LEMMA. *If $\sum_{k=0}^{n} a_k x^k \leqq 1$, $0 \leqq x \leqq 1$, then*

$$|a_k| \leqq 2^{2k} \frac{n(n+k-1)!}{(n-k)!(2k)!}.$$

To derive this bound, one demonstrates that

$$T_n(2x-1) = \sum_{k=0}^{n} (-1)^k 2^{2k} \frac{n(n+k-1)!}{(n-k)!(2k)!} x^k$$

is maximal for each coefficient. For proof, see [1, p. 30]. Note that the above upper bounds yield the estimate $|a_k| \leqq 3^{2n}$, $0 \leqq k \leqq n$.

PROOF OF THE THEOREM. By (1), $\rho_n \geqq c_1/n^2$. To prove that $\rho_n \leqq c_2/n^2$, we shall choose an nth degree approximating polynomial in the following manner. Let f be an arbitrary contraction. Let $S_n(x)$ be the nth partial sum of the infinite series, $S(x)$. Consider f to be extended onto the unit square by setting $f(x, y) = f(x, S(x))$. It will be shown that there is some nth degree polynomial, $p_n(x, y)$, such that $|f - p_n| \leqq c/n^2$, c independent of n. This polynomial will be our approximating polynomial on the given curve. The proof will be completed by showing that when passing from one curve to another along a line parallel to the y-axis, the variation of p_n is of smaller order than $1/n^2$.

Let Q_n be the curve $y = S_n(x)$. Any monomial, x^k, $k = 0, 1, \cdots, n^2 + n$, on Q_n is equal to an nth degree polynomial in x and y as follows:

$$x^k = x^k, \qquad k = 0, 1, \cdots, n,$$
$$x^{n+1} = \frac{xy - (a_1x^2 + a_2x^3 + \cdots + a_{n-1}x^n)}{a_n}.$$

Now,

$$x^{n+k} = \frac{x^ky - (a_1x^{k+1} + a_2x^{k+2} + \cdots + a_{n-1}x^{k+n-1})}{a_n}, \quad k = 1, 2, \cdots, n,$$

is a polynomial of degree n in x and of first degree in y by induction on k. Now,

$$(2) \quad x^{mn+k} = \frac{x^{(m-1)n+k}y - (a_1x^{(m-1)n+k+1} + \cdots + a_{n-1}x^{(m-1)n+k+1})}{a_n},$$
$$m = 1, 2, \cdots, n;\ k = 0, 1, \cdots, n,$$

is a polynomial of degree n in x and of degree m in y by induction on m and k.

It follows from the identities (2) that on Q_n any polynomial in x of degree n^2, $\sum_0^{n^2} b_kx^k$, can be expressed as $\sum_0^n c_k(x)y^k$, where $c_k(x)$ is an nth degree polynomial. It is desirable to find an upper bound for $|c_k(x)|$, $k=0, 1, \cdots, n$, $0\leqq x\leqq 1$. From the relations (2), a calculation verifies that

$$n^3\left|\frac{a_{n-1}}{a_n}\right|^{n^2} \max_k |b_k|$$

is an upper bound. If $\left|\sum_0^{n^2} b_kx^k\right| \leqq A$, we have, by the lemma, and from the inequality

$$|a_{n-1}|^{n^2} < O(1/n^7 3^{2n^2})$$

the estimate

$$(3) \qquad \max_{k,x} |c_k(x)| \leqq A/2n^4 a_n^{n^2}.$$

Now, let f be an arbitrary contraction extended on Q_n as described. Let $g(x)=f(x, S_n(x))$, $0\leqq x\leqq 1$. If $|g'(x)|<M$, then $g(x)/M$ is a contraction. (By the hypothesis on $|a_n|$, $M<1+1/(1-|a_2|)$.) Hence, by Jackson's theorem, there is an n^2-degree polynomial, $p_{n^2}(x)$, such that

$$|g(x) - p_{n^2}(x)| \leqq Mc/n^2, \qquad 0 \leqq x \leqq 1,$$

where c is the constant of Jackson's theorem. Let $p_n(x, y)$ be the nth degree polynomial gotten from $p_{n^2}(x)$ by the relations (2). For any $x\in[0, 1]$,

$$
\begin{aligned}
&|f(x,S(x)) - p_n(x,S(x))| \leqq |f(x,S(x)) - f(x,S_n(x))| \\
&\quad + |f(x,S_n(x)) - g(x,0)| + |g(x,0) - p_{n^2}(x,0)| \\
&\quad + |p_{n^2}(x,0) - p_{n^2}(x,S_n(x))| + |p_{n^2}(x,S_n(x)) - p_n(x,S_n(x))| \\
&\quad + |p_n(x,S_n(x)) - p_n(x,S(x))|.
\end{aligned} \tag{4}
$$

By Taylor's theorem,

$$\max_{x\in[0,1]} |p_n(x,S_n(x)) - p_n(x,y)| \leqq \max_{x\in[0,1]} \sum_k \left| \frac{d^k}{dy^k} p_n(x,y)(y - S_n(x))^k \right| \frac{1}{k!} \cdot$$

Since, in particular, for $y = S(x)$ we have

$$|y - S_n(x)| < 2|a_{n+1}|,$$

(3) gives us

$$|p_n(x,S_n(x)) - p_n(x,S(x))| \leqq A/n^2.$$

We have already established that the third term on the right-hand side of (4) is bounded by Mc/n^2, while the first, second, fourth and fifth terms are zero by definition. Hence,

$$|f(x,S(x)) - p_n(x,S(x))| < (Mc + A)/n^2. \qquad \text{Q.E.D.}$$

In Theorem 1, we have considered approximation to the class of contractions, K. This theorem will now be extended to any class of functions equicontinuous on Q. Let P be defined as before. Let C be the class of all functions in $C(Q)$, whose modulus of continuity is $\leqq \omega(\delta)$ (where $\omega(\delta)$ is the modulus of continuity of some continuous function) and define

$$\rho^*(P) = \max_{f\in C} \min_{p\in P} \max_{x\in Q} |f - p|.$$

THEOREM 2. *Let Q be the curve*

$$y = \sum_{k=1}^{\infty} a_k x^k, \qquad 0 \leqq x \leqq 1,$$

where $|a_{n+1}| < |a_n|^{n^2}$ and $|a_1| \leqq 1$. Then there exist positive constants, c_1, c_2, such that

$$c_1\omega(1/n^2) \leqq \overset{*}{\rho}_n \leqq c_2\omega(1/n^2).$$

Let Q be a simple normal curve. For z, $w \in Q$, denote by $d(z, w)$ the length of that portion of Q connecting z to w. We call Q *properly normal* if there exists M such that $d(z, w) \leqq M|z - w|$.

Theorem 2 follows immediately from Theorem 1 and the following

LEMMA. *If Q is a properly normal curve of length* 1, *then*

$$\overset{*}{\rho}_n(P) < 4M\omega(\rho_n(P)).$$

PROOF. Let $f \in C$. Let z_1 be an endpoint of Q. By hypothesis, $0 \leqq d(z_1, z) \leqq 1$. Partition Q into $[1/\rho]$ curves, Q_k, $k = 1, 2, \cdots, [1/\rho]$, of equal length. Suppose, first, that $1/\rho$ is an integer. Consider $L(z)$, the function equal to $f(z)$ at the endpoints, z_k, z_{k+1}, of the Q_k, and defined "linearly" in between; i.e.,

$$L(z) = \frac{(z - z_k)f(z_{k+1}) - (z_{k+1} - z)f(z_k)}{z_{k+1} - z_k}, \qquad z \in Q_k.$$

We have, then, $|L(z_{k+1}) - L(z_k)| \leqq \omega(|z_{k+1} - z_k|) \leqq \omega(\rho)$. Hence,

$$|L(z) - L(w)| \leqq \omega(\rho)L(z, w)/\rho \leqq M\omega(\rho)|z - w|/\rho,$$

implying that $\rho L(z)/M\omega(\rho)$ is a contraction and, thus, that there exists $p_1(z)$ such that

$$|\rho L(z)/M\omega(\rho) - p_1(z)| < \rho.$$

Letting $p_2(z) = Mp_1(z)\omega(\rho)/\rho$, we have

$$|L(z) - p_2(z)| \leqq M\omega(\rho).$$

Now,

$$\begin{aligned} |f(z) - p_2(z)| \leqq\ & |f(z) - f(z_k)| + |f(z_k) - L(z_k)| \\ & + |L(z_k) - L(z)| + |L(z) - p_2(z)| \leqq 3M\omega(\rho). \end{aligned}$$

We take care of the case where $1/\rho$ is not an integer by weakening our estimate to $4M\omega(\rho)$.

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