ON NORMS OF IDEMPOTENT MEASURES

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Let G be a locally compact abelian group and Γ its dual. The Fourier transform of a measure μ on G is the function $\hat{\mu}$ on Γ defined by

$$\hat{\mu}(\gamma) = \int_G (-x, \gamma) d\mu(x) \qquad (\gamma \in \Gamma).$$

If μ is idempotent, then $\hat{\mu}^2 = \hat{\mu}$, so that $\hat{\mu}(\gamma) = 1$ or 0 for all $\gamma \in \Gamma$. Define

$$S(\mu) = \{\gamma \in \Gamma : \hat{\mu}(\gamma) = 1\}.$$

As is stated in [1], every idempotent measure μ on G has norm 1 if and only if $S(\mu)$ is a coset of a subgroup of Γ, and if μ has norm greater than 1, then $\|\mu\| \geqq 5^{1/2}/2 \sim 1.118$. If G is compact and if $d\mu(\chi) = [1 + (\chi, \gamma)] dm(\chi)$ for some $\gamma \in \Gamma$, with order greater than 2, where m is the Haar measure of G normalized so that $\|m\| = 1$, then

$$\|\mu\| = \int_G |1 + (\chi, \gamma)| \, dm(\chi) \geqq \frac{1 + 2^{1/2}}{2} \sim 1.207,$$

the value $(1 + 2^{1/2})/2$ being attained when γ has order 4. The purpose of this paper is to show that this constant is best possible.

THEOREM. *If μ is an idempotent measure on G with $\|\mu\| > 1$, then $\|\mu\| \geqq (1 + 2^{1/2})/2$.*

PROOF. Since the support group of any idempotent measure is compact, it is sufficient to prove the theorem for compact G. Without loss of generality, we may also assume that $\hat{\mu}(0) = 1$, i.e., that $S(\mu)$ contains the identity 0 in Γ. So, hereafter, we shall assume that G is compact and $0 \in S(\mu)$.

Since $\|\mu\| > 1$, $S(\mu)$ is not a coset in Γ, in particular, not a subgroup. Therefore we can find γ_0, γ_1 in $S(\mu)$ so that $\gamma_1 - \gamma_0 \notin S(\mu)$.

First assume $2\gamma_0 \in S(\mu)$, and put

$$\begin{aligned} f(\chi) &= (-\chi, \gamma_0)[1 + \mathrm{Re}(-\chi, \gamma_0)] + (-\chi, \gamma_1)[1 - \mathrm{Re}(-\chi, \gamma_0)] \\ &= (-\chi, \gamma_0) + (-\chi, \gamma_1) + \tfrac{1}{2}[(-\chi, 2\gamma_0) + 1] \\ &\quad - \tfrac{1}{2}[(-\chi, \gamma_1 + \gamma_0) + (-\chi, \gamma_1 - \gamma_0)]. \end{aligned}$$

The first expression for f shows that

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$$|f(\chi)| \leqq |1 + \mathrm{Re}(-\chi, \gamma_0)| + |1 - \mathrm{Re}(-\chi, \gamma_0)| = 2,$$

and the second expression for f shows that

$$\int_G f d\mu = 3 - \tfrac{1}{2}\hat{\mu}(\gamma_1 + \gamma_0) \geqq 2.5.$$

Hence, $\|\mu\| \geqq \frac{1}{2}|\int_G f d\mu| \geqq 1.25$.

Next, assume that $2\gamma_0 \notin S(\mu)$. Let Γ_0 be the cyclic group generated by γ_0, m_0 the Haar measure of the annihilator of Γ_0, and μ_0 the convolution of μ and m_0. Then μ_0 is idempotent and

$$\|\mu_0\| = \|\mu * m_0\| \leqq \|\mu\| \cdot \|m_0\| = \|\mu\|.$$

Hence it will suffice to show $\|\mu_0\| \geqq (1+2^{1/2})/2$. Note that $S(\mu_0) = \Gamma_0 \cap S(\mu)$ and that $S(\mu_0)$ contains the elements 0 and γ_0 but does not contain $2\gamma_0$.

Now at least one of the following conditions is satisfied for $S(\mu_0)$:

(a) $S(\mu_0)$ is a union of two cosets of a subgroup Λ of Γ_0.

(b) For some integer p, $S(\mu_0)$ contains the elements $p\gamma_0$, $(p+1)\gamma_0$, and $(p+2)\gamma_0$.

(c) For some integer q, $S(\mu_0)$ contains $q\gamma_0$ but neither $(q+1)\gamma_0$ nor $(q-1)\gamma_0$.

(d) None of the conditions (a), (b), and (c) holds.

Suppose $S(\mu_0)$ satisfies (a). Then it is obvious that $S(\mu_0)$ is the union of two cosets Λ and $(\gamma_0+\Lambda)$ since $2\gamma_0 \notin S(\mu_0)$. Hence μ_0 has the form

$$d\mu_0(\chi) = [1 + (\chi, \gamma_0)]dm_1(\chi),$$

where m_1 denotes the Haar measure of the annihilator of Λ, and so $\|\mu_0\| \geqq (1+2^{1/2})/2$.

If the condition (b) is satisfied, there exists an integer q such that the elements $q\gamma_0$, $(q+1)\gamma_0$, and $(q+2)\gamma_0$ belong to $S(\mu_0)$ but either $(q-1)\gamma_0$ or $(q+3)\gamma_0$ does not belong to $S(\mu_0)$. If $(q-1)\gamma_0 \notin S(\mu_0)$, put

$$f(\chi) = (-\chi(q+1)\gamma_0)[1 + \mathrm{Re}(-\chi, \gamma_0)] + (-\chi, q\gamma_0)[1 - \mathrm{Re}(-\chi, \gamma_0)].$$

We have $\|\mu_0\| \geqq \frac{1}{2}|\int_G f d\mu_0| = 2.5/2 = 1.25$. If $(q+3)\gamma_0 \notin S(\mu_0)$, we can also define a function f on G such that $|f(\chi)| \leqq 2$ and $\int_G f d\mu_0 = 2.5$, and so $\|\mu_0\| \geqq 1.25$.

Suppose that condition (c) is the case. Define a function f on G by

$$f(\chi) = [1 + \mathrm{Re}(-x, \gamma_0)] + (-\chi, q\gamma_0)[1 - \mathrm{Re}(-\chi, \gamma_0)].$$

We have again $\|\mu_0\| \geqq \frac{1}{2}|\int_G f d\mu_0| \geqq 2.5/2 = 1.25$.

Finally, suppose that the condition (d) holds. This implies that

$S(\mu_0)$ is "nonperiodic" and that if $(p-1)\gamma_0 \notin S(\mu_0)$ and $p\gamma_0 \in S(\mu_0)$ then $(p+1)\gamma_0 \in S(\mu_0)$ and $(p+2)\gamma_0 \notin S(\mu_0)$. In this case we may clearly assume (if necessary, replace γ_0 by $-\gamma_0$ and translate $S(\mu_0)$ by $p\gamma_0$ for some integer p) that there exists an integer $q \geqq 3$ such that $q\gamma_0 \in S(\mu_0)$, $(q+1)\gamma_0 \in S(\mu_0)$ but $-p\gamma_0 \notin S(\mu_0)$ whenever $1 \leqq p < q$. Put

$$f(\chi) = [1 + \mathrm{Re}(-\chi, (q+1)\gamma_0)] + (-\chi, \gamma_0)[1 - \mathrm{Re}(-\chi, (q+1)\gamma_0)].$$

Since $(q+2)\gamma_0 \notin S(\mu_0)$ by our assumption, we have

$$\int_G f d\mu_0 = 2.5 + \tfrac{1}{2}[\hat{\mu}_0(-(q+1)\gamma_0) - \hat{\mu}_0(-q\gamma_0)].$$

Thus the right side in the above equality is greater than or equal to 2.5 if $\hat{\mu}_0(-(q+1)\gamma_0) - \hat{\mu}_0(-q\gamma_0) \geqq 0$. But $-q\gamma_0 \in S(\mu_0)$ implies $-(q+1)\gamma_0 \in S(\mu_0)$ since $-(q-1)\gamma_0 \notin S(\mu_0)$. Therefore $\hat{\mu}_0(-(q+1)\gamma_0) - \hat{\mu}_0(-q\gamma_0) \geqq 0$ in any case, and we have $\int_G f d\mu_0 \geqq 2.5$, and $\|\mu_0\| \geqq 1.25$.

The proof of the theorem is now completed.

Reference

1. W. Rudin, *Fourier analysis on groups*, Interscience, New York, 1962.

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