ON THE METASTABLE HOMOTOPY OF $O(n)$

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The purpose of this note is to present an alternate proof of the main result of [3]. In particular, we prove

THEOREM 1 (1 OF [3]). *If $k>4$, a nontrivial stable real vector bundle over S^{4k} is the sum of an irreducible $(2k+1)$-plane bundle and a trivial bundle.*

This result has several geometric applications and easily implies

THEOREM 2 (2 OF [3]). *For $q<2(n-1)$ and $n\geqq 13$,*

$$\pi_q(O(n)) = \pi_q(O) \oplus \pi_{q-1}(V_{2n,n}).$$

We will be only concerned with the proof of Theorem 1 in this note. Extensive calculation of $\pi_{q-1}(V_{2n,n})$ are given in [4].

The proof will be preceded by several lemmas which use the following notation. Let X be a space. The symbol $X[k]$ means the $(k-1)$-connected fibering over X. $E_r^{s,t}(X)$ is the rth term in the Adams spectral sequence [1] for X leading to the group associated with ${}_2\pi_*^S(X)$. Thus $E_2^{s,t}(X)=\mathrm{Ext}_A^{s,t}(\tilde{H}^*(X;Z_2),Z_2)$, where A is the mod 2 Steenrod algebra. V_k is the fiber of $BSO_k\to BSO$.

LEMMA 1. *If $t-s\leqq 4k$ and $k>4$, then*

$$E_2^{s,t}(BSO_{2k+1}[2k+1]) \simeq E_2^{s,t}(BSO[2k+1]) \oplus E_2^{s,t}(V_{2k+1}).$$

PROOF. Let p be the smallest integer such that $2^p>4k$. For $p=4a+b$, $0\leqq b\leqq 3$, let $j(p)=8a+2^b$. Let $i_p: BSO[j(p)]\to BSO$ be the usual inclusion. Then $i_p^*w_j=0$ for all $j<2^p$ [5]. If $k>4$, then $BSO_{2k+1}[j(p)]$ is the total space of $i_p^!\gamma_{2k+1}$ where γ_{2k+1} is the bundle $BSO_{2k+1}\to BSO$. Therefore

$$H^q(BSO_{2k+1}[j(p)]) \simeq \sum_{u+v=q} H^u(BSO[j(p)]) \oplus H^v(V_{2k+1})$$

for $q\leqq 4k+1$ as Z_2 modules. Since i': $BSO[2k+1]\to BSO[j(p)]$ induces the zero map in cohomology if $2k+1>j(p)$ [5], we have

$$H^q(BSO_{2k+1}[2k+1]) \simeq \sum_{u+v=q} H^u(BSO[2k+1]) = H^v(V_{2k+1})$$

as A modules. This implies the lemma.

Received by the editors March 1, 1967.

[1] This work was supported in part by the U. S. Army Research Office (Durham).

COROLLARY 2. *The projection* $p : BSO_{2k+1}[2k+1] \to BSO[2k+1]$ *induces an epimorphism*

$$p_{\#}: E_2^{s,t}(BSO_{2k+1}[2k+1]) \to E_2^{s,t}(BSO[2k+1]).$$

LEMMA 3. *Let* $\omega_{4k}: S^{4k} \to BSO[2k+1]$ *be a generator of* $\pi_{4k}(BSO[2k+1])$.

$$\begin{aligned} \omega_{4k} \text{ has filtration } &\geqq k-1, && k \equiv 0, 1 \pmod 4, \\ &\geqq k-2, && k \equiv 2 \pmod 4, \text{ and} \\ &\geqq k, && k \equiv 3 \pmod 4) \end{aligned}$$

in the Adams spectral sequence sense.

PROOF. Consider the diagram

$$S^{4k} \xrightarrow{i_f} BSO[4k] \to \cdots \xrightarrow{i_1} BSO[2k \times 1].$$

In cohomology each map i_u^* is zero if the connectivity increases. Maps between spaces which induce the zero map in cohomology have filtration $\geqq 1$. Thus the question of filtration reduces to only counting the number of nonzero homotopy groups between $2k+1$ and $4k$ in BSO.

LEMMA 4. $E_2^{s,t}(V_{2k+1}) = 0$ *if* $t-s = 4k-1$ *and*

$$\begin{aligned} s &\geqq k, && k \equiv 0, 1 \ (2) \\ &\geqq k+1, && k \equiv 2, 3 \ (4). \end{aligned}$$

PROOF. Let $0 \leqq u < 8$ be such that $2k+1+u \equiv 4k-1 \pmod 8$. Let $0 \leqq v < 4$ be such that $k \equiv v \pmod 4$. It is an easy calculation to compute $\mathrm{Ext}_A^{s,t}(\tilde{H}^*(V_{2k+1}), Z_2)$ by minimal resolution [1] for $t = 2k+1 +u+s$ and $s = v, v+1$ (or one can use the tables in [4]). The Adams periodicity theorems [2] enable one to use this calculation to prove the lemma.

Now we can prove the main theorem if $k \not\equiv 2 \pmod 4$.

PROOF OF THEOREM 1. Let $\bar{\alpha} \in E_2^{s,t}(BSO_{2k+1}[2k+1])$ project to the class in $E_2^{s,t}(BSO[2k+1])$ to which $[\omega_{4k}]$ projects. We are finished if we show that $\bar{\alpha}$ is a permanent cycle. Since $p_{\#}\bar{\alpha}$ is not a boundary, $\bar{\alpha}$ is not a boundary. If $\delta_r \bar{\alpha} = \beta$, then β must be in the summand $E_2^{s,t}(V_{2k+1})$ but Lemma 4 says that this group is zero for all possible s and t values that β might have.

To finish the proof of Theorem 1, we just need to indicate how to fix the argument for $k \equiv 2$ (4). Let $\alpha_i \in H^{u(i)}(V_{2k+1})$ be a generating set over A. Let

$$BSO_{2k+1}[j(p)] \xrightarrow{f} BSO[j(p)] \times \prod_i K(Z_2, u(i))$$

be the usual map on the first factor, and on the second factor $f^*(\iota_i) = \alpha_i$ where ι_i is the characteristic class of $K_i = K(Z_2, u(i))$. Let F be the fiber of f and let $\bar{p}: E \to BSO[2k+1]$ be the fiber space induced from f by the composite map

$$BSO[2k+1] \to BSO[j(p)] \to BSO[j(p)] \times \prod_i K_i,$$

where the maps are the obvious ones. Following the arguments given above we can prove

LEMMA 5. *If* $t-s \leqq 4k$ *and* $k>4$, *then*:
(i) $E_2^{s,t}(E) \simeq E_2^{s,t}(BSO[2k+1]) \oplus E_2^{s,t}(F)$;
(ii) $E_2^{s,t}(F) \simeq E_2^{s+1,t+1}(V_{2k+1})$.

The theorem follows from this as above.

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