ON THE SIZE OF THE BRAUER GROUP

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It is well known that the isomorphism classes of finite-dimensional central division algebras over a field K form a group $B(K)$, called the Brauer group of K. This group may be described cohomologically as follows: let G_K be the Galois group of the separable closure K_s of K, then $B(K)=H^2(G_K, K_s^*)$ where the cohomology is that of profinite groups [**3**]. This group is an important arithmetic invariant which has been studied in various contexts.

An examination of known cases has led us to the conjecture that if $B(K)$ has finite exponent, then either the exponent is two or $B(K)=(0)$. A somewhat stronger statement seems to be true. We write A_p for the p-primary component of any abelian group A.

CONJECTURE. Either $B(K)_p$ contains a nontrivial divisible subgroup or $2B(K)_p=(0)$. In particular, if p is odd, the second alternative implies $B(K)_p=(0)$.

In another publication [**1**] Auslander and Brumer have shown the conjecture is true for function fields in $n\geqq 1$ variables over arbitrary ground fields. The purpose of this note is to provide some additional evidence for the conjecture based on the properties of pro-p-groups.

LEMMA 1. *If p is the characteristic of k, then $B(K)$ is divisible by p.*

PROOF. In [**3**, Chapter II] it is shown that $\text{cd}_p\, G_K\leqq 1$. Consider the short exact sequence

$$(1)\to K_s^* \xrightarrow{p} K_s^* \to K_s^*/K_s^{*p}\to (1).$$

We must have $H^2(G_K, K_s^*/K_s^{*p})=(0)$. Thus the map

$$B(K)\xrightarrow{p} B(K)$$

is onto.

This lemma shows that for our problem we may assume p is different from the characteristic of K. We do so from now on.

Let μ be the group of all p^nth roots on unity $n=1, 2, 3, \cdots$ contained in K_s.

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LEMMA 2. (a) $B(K)_p \approx H^2(G_K, \mu)$. (b) *For each subgroup H of G_K, $H^1(H, \mu)$ is divisible.*

PROOF. The group K_s^*/μ is uniquely p divisible, hence the p-primary part of its cohomology vanishes in dimensions greater than 0. The short exact sequence $(1) \to \mu \to K_s^* \to K_s^*/\mu \to (1)$ thus implies the truth of the first assertion.

Let μ_p be the group of pth roots of unity and consider the commutative diagram.

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \mu_p & \to & \mu & \xrightarrow{p} & \mu & \to & 1 \\
 & & \downarrow = & & \downarrow & & \downarrow & & \\
1 & \to & \mu_p & \to & K_s^* & \xrightarrow{p} & K_s^* & \to & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K_s^*/\mu & \xrightarrow{\approx} & K_s^*/\mu & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

In cohomology we have

$$\begin{array}{ccccccc}
H^1(H, \mu) & \xrightarrow{p} & H^1(H, \mu) & \to & H^2(H, \mu_p) & \to & H^2(H, \mu) \\
\downarrow & & \downarrow & & \downarrow = & & \downarrow \approx \\
H^1(H, K_s^*) & \to & H^1(H, K_s^*) & \to & H^2(H, \mu_p) & \to & H^2(H, K_s^*)_p.
\end{array}$$

By Hilbert's Theorem 90, $H^1(H, K_s^*) = (0)$. The second assertion follows at once.

LEMMA 3. *Suppose G is a pro-nilpotent group and P its p-Sylow subgroup, then either $H^2(G, \mu) \approx H^2(P, \mu)$ or $H^2(G, \mu) = (0)$.*

PROOF. Since G is pro-nilpotent, P is a direct factor of G, $G = PN$. The order of N is prime to p. Thus $H^m(N, \mu) = (0)$ for all $m > 0$. From the Hochschild-Serre spectral sequence, we see $H^2(P, \mu^N) \approx H^2(G, \mu)$. The automorphism group of μ is isomorphic to U, the group of p-adic units. From this and the fact that the order of N is prime to p, one sees that μ^N is either (1) or μ. The conclusion follows.

LEMMA 4. *Suppose $G = G_K$ is an abelian pro-p-group and that $\sqrt{(-1)} \in K$ if $p = 2$. Then one of the following must hold.*

(i) *G acts trivially on μ.*

(ii) *G is isomorphic to the p-adic integers, i.e.* cd $G = 1$.

PROOF. A torsion element of G will define a subfield of finite codimension in the algebraic closure of K. Under our hypotheses the well-known theorem of Artin-Schreier shows there can be no such subfield. Thus G is torsion free.

As has already been mentioned, the automorphism group of μ is isomorphic to the p-adic units. G is a pro-p-group. Hence the image of G in $\mathrm{Aut}(\mu)$ is either trivial or isomorphic to the p-adic integers. If neither (i) nor (ii) hold, we can find a subgroup $H=\langle\sigma, \tau\rangle$ of G such that σ acts trivially on μ and τ acts nontrivially. The Hochschild-Serre spectral sequence yields

$$(0) \to H^1(\langle\tau\rangle, \mu) \to H^1(H, \mu) \to H^1(\langle\sigma\rangle, \mu)^{\langle\tau\rangle} \to H^2(\langle\tau\rangle, \mu).$$

The last term vanishes since $\mathrm{cd}\langle\tau\rangle = 1$. Since σ acts trivially on μ, $H^1(\langle\sigma\rangle, \mu) = \mathrm{Hom}(\langle\sigma\rangle, \mu) \approx \mu$. Hence the next to last term is isomorphic to the group of elements of μ left fixed by τ. This group contains at least μ_p but is finite since τ acts nontrivially.

By Lemma 2, $H^1(H, \mu)$ is divisible. We have now arrived at a contradiction since the epimorphic image of a divisible group cannot be a nontrivial finite group.

LEMMA 5. *Let F be a free pro-p-group and R a closed normal subgroup. If $R \neq (1)$, then $R/[R, F]$ is not a torsion group.*

PROOF. Let $F_0 = F$ and define $F_{k+1} = [F_k, F]$. It is known that F_k/F_{k+1} is torsion free (cf. for instance Proposition 2 of [2]). Since $\bigcap F_k = (1)$ our hypotheses imply that for some k we have $R \subset F_k$ but $R \not\subset F_{k+1}$. Since $[R, F] \subset F_{k+1}$, the natural map yields a nontrivial homomorphism

$$R/[R, F] \to F_k/F_{k+1},$$

F_k/F_{k+1} is torsion free, so $R/[R, F]$ cannot be torsion.

THEOREM. *Let K be a field containing all p^nth roots of unity, $n = 1, 2, 3, \cdots$. Let L be the maximal p extension of K, and let P be the Galois group of L over K. Then one of the following must hold:*

(i) *$B(K)_p$ contains a nontrivial divisible subgroup.*

(ii) *P is a free pro-p-group, so that $B(L/K) = (0)$.*

PROOF. Consider the subgroup $B(L/K) = H^2(P, L^*)$ of $B(K)$ consisting of those elements split by L. Since $\mu \subset K^*$ and L has no p-extensions, we can apply the considerations of Lemma 2 to conclude that $B(L/K) \approx H^2(P, \mu)$ and $H^1(P, \mu)$ is divisible. P acts trivially on μ, so $H^1(P, \mu) \approx (P/P_1)\hat{}$, the character group of P/P_1. Pontryagin duality implies P/P_1 is torsion free.

Represent P as the quotient F/R of a free pro-p-group F by a closed normal subgroup R. We may suppose F is minimal (cf. [**3**]). The Hochschild-Serre sequence gives

$$(0) \to H^1(P, \mu) \overset{\alpha}{\to} H^1(F, \mu) \to H^1(R, \mu)^P \to H^2(P, \mu) \to H^2(F, \mu).$$

The last term is zero because cd $F=1$. α is an isomorphism because F was chosen to be minimal. Also, notice that $H^1(R, \mu)^P = \mathrm{Hom}(R/[R, F], \mu)$. Thus

$$\mathrm{Hom}(R/[R, F], \mu) \approx H^2(P, \mu).$$

If P is not free, $R \neq (1)$, and Lemma 5 shows that $R/[R, F]$ is not torsion. Pontryagin duality now shows that $H^2(P, \mu)$ contains a nontrivial divisible subgroup (dual to the maximal torsion-free quotient of $R/[R, F]$!).

COROLLARY 1. *Let K be a field whose Galois group G_K is abelian. The $B(K)_p$ is divisible unless $p=2$ and $\sqrt{(-1)} \notin K$. In the latter case $2B(K)_2$ is divisible.*

PROOF. Lemma 3 implies we may assume G_K is a pro-p-group. Assume that all the p^nth roots of unity are in K. Then, as in the proof of the theorem, $G_K \approx F/F_1$ for some free pro-p-group F. Hence $B(K)_p$ is dual to $F_1/[F_1, F] = F_1/F_2$ which is torsion free. It follows that $B(K)_p$ is divisible.

By Lemma 4 the corollary is proven unless $p=2$ and $\sqrt{(-1)} \notin K$.

Let $L = K(\sqrt{(-1)})$. Then $B(L)_2$ is divisible. Since G_L is of index 2 in G_K, we have the maps res: $B(K) \to B(L)$ and cor: $B(L) \to B(K)$ arising from cohomology. The composition cor $\circ$ res = multiplication by 2. Thus

$$B(K)_2 \supset \mathrm{cor}\, B(L)_2 \supset \mathrm{cor} \circ \mathrm{res}\, B(K)_2 = 2B(K)_2.$$

From the fact that a homomorphic image of a divisible group is divisible, it follows readily that $2B(K)_2$ is divisible.

COROLLARY 2. *Assume all the p^nth roots of unity are in the ground field K, and that G_K is pro-nilpotent. Then either $B(K)_p = (0)$ or $B(K)_p$ contains a nontrivial divisible subgroup.*

PROOF. Let P be a p-Sylow subgroup of G_K. $B(K)_p = H^2(G_K, \mu)$, and by Lemma 3 $H^2(G_K, \mu)$ is either (0) or $H^2(P, \mu)$. Suppose the latter case holds. Since P is a direct factor of G_K, it is isomorphic to the Galois group of the maximal p-extension of K. Thus either $H^2(P, \mu) = (0)$ or it contains a nontrivial divisible subgroup.

Bibliography

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