WHYBURN'S CONJECTURE FOR C^2 MAPS

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In [6] Whyburn made the following conjecture and verified it for the case $n=2$: Suppose f is a light open mapping of one n-cell onto another. If f maps the boundary homeomorphically, then f is a homeomorphism.

If $f\in C^n$, the truth of the conjecture follows from the structure theorem of P. T. Church [1]. The question is still open in the general case. The purpose of this note is to affirm the conjecture for $f\in C^2$; however, it is not required that f be light.

DEFINITIONS. For $f\colon X\to Y$, X and Y topological spaces, f is *open* if $f(U)$ is open for every U open in X; f is *light* if $f^{-1}(p)$ is totally disconnected for every $p\in f(X)$. Let B_f denote the set of all $x\in X$ such that f is not a local homeomorphism at x. If X is an open subset of E^n, Euclidean n-space, and Y is open in E^m, let $R_j(f)$ be the set of all $x\in X$ such that the rank of the Jacobian matrix of f at x is less than or equal to j, $0\leqq j\leqq m$. Let D^n denote the closed n-cell and let S^{n-1} denote its boundary, the $n-1$ sphere.

By $f\in C^n$ on D^n we mean that there exists an open set $U\supset D^n$ and a function $F\colon U\to E^n$ such that $F|D^n=f$ and F has continuous second order partial derivatives in U.

THEOREM. *Let $F\colon D^n\to D^n$ be an open C^2 mapping such that $f^{-1}(S^{n-1})=S^{n-1}$ and $f|S^{n-1}$ is a homeomorphism. Then f is a homeomorphism.*

PROOF. First we observe that $B_f\subset R_{n-2}(f)$ [1, Theorem 1.4, p. 89]. (The theorem is stated for E^n but works equally well for D^n.) Let $P\colon E^n\to E^{n-1}$ be defined by $P(x_1,\cdots,x_n)=(x_1,\cdots,x_{n-1})$ and let $g=P\circ f$. Clearly $R_{n-2}(f)\subset R_{n-2}(g)$. By Sard's theorem [3], since $g\in C^2$, the $n-1$ dimensional measure of $g(R_{n-2}(g))$ is zero; thus this is also true of $g(B_f)$.

Let $W=\{y\in D^n|\operatorname{card} f^{-1}(y)\geqq 2\}$. We will be done if we show that $W=\emptyset$. Suppose then that $W\neq\emptyset$; since f is open, W is a nonempty open set and so is $P(W)$. Now $g(B_f)$ contains no open set, so there exists $z\in P(W)-g(B_f)$. Then $P^{-1}(z)$ is a line intersecting S^{n-1} and W, but containing no point of $f(B_f)$. Pick $a\in P^{-1}(z)\cap S^{n-1}$ and $b\in P^{-1}(z)\cap W$. By hypothesis, $\operatorname{card} f^{-1}(a)=1$ and by definition of W, $\operatorname{card} f^{-1}(b)>1$. However, by Theorem 6.1 [5, p. 199], $\operatorname{card} f^{-1}(a)=\operatorname{card} f^{-1}(b)$. This contradiction proves the theorem.

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REMARKS. If we replace the hypothesis that $f \in C^2$ by the condition that for each $x \in D^n$, $f^{-1}(f(x))$ is finite, then the theorem is true by [**4**, Theorem 5.5, p. 8].

Cronin and McAuley [2] have proved the theorem assuming $f \in C^1$ and an additional hypothesis holds. If f is only assumed C^1, it is not known if the theorem is true.

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