THE BORDISM CLASS OF A QUASI-SYMPLECTIC MANIFOLD

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It is not known which unoriented bordism classes contain Sp-manifolds (the stable tangent bundle admitting a reduction to the symplectic group). As an approximation, a class of smooth manifolds is introduced, containing all Sp-manifolds and quaternionic projective spaces $HP(n)$, and its image in the unoriented bordism ring $\mathfrak{N}$ determined.

Let $\tau(M)$ denote the tangent bundle of a smooth manifold. If ξ is a right quaternionic vector bundle, the conjugate ξ^* is a left quaternionic vector bundle; thus for right quaternionic vector bundles ξ and η, the tensor product $\xi \otimes_H \eta^*$ is a real vector bundle.

A closed manifold M will be said to be *quasi-symplectic* if for some k and finite collection of right quaternionic vector bundles ξ_i, η_i

$$\tau(M) \oplus kR = \sum \xi_i \otimes_H \eta_i^*.$$

Since $\xi \otimes_H H^* = \xi_R$ is the underlying real bundle of ξ, an Sp-manifold is quasi-symplectic. It is clear that $M \times M'$ is quasi-symplectic if M and M' are.

THEOREM. *An unoriented bordism class $[M]_2 \in \mathfrak{N}$ contains a quasi-symplectic manifold if and only if $[M]_2$ is a fourth power in $\mathfrak{N}$.*

According to Milnor [3], a class $[M]_2 \in \mathfrak{N}$ is a fourth power if and only if all Stiefel-Whitney numbers of M involving some w_i with $i \neq 0 \pmod 4$ vanish. Moreover, if $[M]_2 = ([N]_2)^4$ then $w_{4i_1} \cdots w_{4i_r}[M] = w_{i_1} \cdots w_{i_r}[N]$. Thus in one direction the theorem follows from

LEMMA 1. *If ξ and η are right quaternionic vector bundles, then $w_i(\xi \otimes_H \eta^*) = 0$ for $i \neq 0 \pmod 4$.*

PROOF. By the splitting principle, ξ and η may be taken to be quaternionic line bundles. In this case, it follows by the methods of [2] or [4] that

$$w(\xi \otimes_H \eta^*) = 1 + w_4(\xi_R) + w_4(\eta_R). \tag{1}$$

Moreover, $w(\xi \otimes_H \xi^*) = 1$ for ξ a quaternionic line bundle; and for any

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right quaternionic vector bundle η, $w(\eta_R)$ is the mod 2 reduction of the total (symplectic) Pontrjagin class $p(\eta)$.

Before returning to the proof of the theorem, we prove

LEMMA 2. *If M is quasi-symplectic and $\xi \to M$ is a right quaternionic vector bundle, then the associated quaternionic projective space bundle $P(\xi)$ is quasi-symplectic.*

PROOF. Let π: $P(\xi) \to M$ denote the projection, and $\eta \to P(\xi)$ the canonical symplectic line bundle over $P(\xi)$. It follows from [4, Theorem 1.3] and [2] that

$$\tau(P(\xi)) \oplus \eta \otimes_H \eta^* = \pi^*\tau(M) \oplus \pi^*\xi \otimes_H \eta^*, \tag{2}$$

hence $P(\xi)$ is quasi-symplectic if M is. In particular, $HP(n)$ is quasi-symplectic (take M a point) and

$$\tau(HP(n)) \oplus \eta \otimes_H \eta^* = (n + 1)\eta_R \tag{3}$$

as is well known.

We now complete the proof of the theorem. Let $\eta \to RP(n)$ denote the Hopf line bundle and put $RP(m, n) = P(\eta \oplus mR)$; thus $RP(m, n)$ is fibred over $RP(n)$ with fibre $RP(m)$. According to P. Anderson [1], the unoriented bordism ring $\mathfrak{N}$ is generated by the classes $[RP(m, n)]_2$. In a similar fashion, let $HP(m, n) = P(\zeta \oplus mH)$ with $\zeta \to HP(n)$ the canonical symplectic line bundle. By Lemma 2, $HP(m, n)$ is quasi-symplectic. In view of (1), (2) and (3), it follows readily as in [1] that

$$w_{4i_1} \cdots w_{4i_r}[HP(m, n)] = w_{i_1} \cdots w_{i_1}[RP(m, n)],$$

hence $[HP(m, n)]_2 = ([RP(m, n)]_2)^4$. Since $\mathfrak{N}$ is generated by the $[RP(m, n)]_2$, every class $([M]_2)^4$ of $\mathfrak{N}$ contains a quasi-symplectic manifold. Q.E.D.

REMARK. Consider quasi-symplectic manifolds M^k such that the normal bundle to M^k has the form $\eta \otimes_H \eta^* \oplus \zeta_R$ for a quaternionic line bundle η and quaternionic bundle ζ. There result bordism groups $'\Omega_k^{Sp}$, and $'\Omega^{Sp} = \sum_k '\Omega_k^{Sp}$ is a graded left module over the symplectic bordism ring Ω^{Sp}. It can be shown that $'\Omega^{Sp}$ is a free module over Ω^{Sp} on the classes $[HP(n)] \in '\Omega_{4n}^{Sp}$, $n \geqq 0$. Thus the image of $'\Omega^{Sp}$ in $\mathfrak{N}$ consists of the sums of classes $\alpha[HP(2n)]_2 = \alpha([RP(2n)]_2)^4$ for $n \geqq 0$ and $\alpha \in \mathfrak{N}$ containing an Sp-manifold. However the image of Ω^{Sp} in $\mathfrak{N}$ is not known.

ADDED IN PROOF. R. E. Stong has shown that every symplectic manifold of positive dimension less than 24 bounds in the unoriented sense. See *Some remarks on symplectic cobordism*, Ann. of Math. **86** (1967), 425–433.

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