SUFFICIENT CONDITIONS FOR A CLOSED SET TO LIE ON THE BOUNDARY OF A 3-CELL

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Let F be a closed subset of a 2-sphere S in S^3, and let V be a component of S^3-S. Using some results and proofs from [7] and [8] we give several conditions that are sufficient for F to lie on the boundary of a 3-cell. One such condition is Property $(*, F, V)$ which is defined in [7]. Property $(*, F, V)$ roughly means that the polyhedral approximation S' to S obtained using Bing's Side Approximation Theorem [2] can be chosen such that S' lies "almost" in V and $S\cap S'$ lies in the union of a finite collection of disjoint small disks in $S-F$.

Property $(*, F, S)$ is defined in [7] to mean that Property $(*, F, V)$ is satisfied for each component V of S^3-S. The author [7] established a conjecture made by Gillman [5] and then used some of Bing's techniques [1] to show that F lies on a tame 2-sphere if Property $(*, F, S)$ is satisfied. Unfortunately the following theorem was not included in [7] and is not a direct consequence of any of the results there.

THEOREM 1. *If* $\epsilon>0$, *F is a closed subset of a 2-sphere S in S^3, and V is a component of S^3-S such that Property* $(*, F, V)$ *is satisfied, then there is a continuum M on S and a null sequence* $\{D_i\}$ *of disjoint ϵ-disks on S such that* (1) $M=S-\bigcup \operatorname{Int} D_i$, (2) *Property* $(*, M, V)$ *is satisfied,* (3) $F\subset M-\bigcup D_i$, *and* (4) *M lies on the boundary of a 3-cell.*

Lister [6] made use of Theorem 1 in showing that $h(F)$ lies on the boundary of a 3-cell if F and S satisfy Property $(*, F, V)$ and h is a homeomorphism of $S\cup V$ into S^3. Burgess and Loveland [4] used Theorem 2, which is a consequence of Theorem 1, in giving conditions for F to lie on the boundary of a 3-cell.

A proof for Theorem 1 can be obtained following the same general pattern as in the proof of Theorem 6 of [7] provided we have a "one-sided" version of Gillman's conjecture to replace Theorem 3 of [7]. Such a version of the conjecture is stated and proved here as Lemma 1. The proof given for Lemma 1 is based on consequences of a "two-sided" version of the conjecture (Theorem 3 of [7]) given in [7]. We feel that to prove Theorem 1 here would be to a large extent a repeat of §2 of [7]. For this reason we prove only Lemma 1 and then we rely on [7] for an indication of how Lemma 1 is used to obtain a proof for Theorem 1.

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LEMMA 1. *Suppose F is a closed subset of a 2-sphere S in E^3 such that Property $(*, F, \text{Int } S)$ is satisfied. There is a positive number ϵ such that if h is a homeomorphism of S into E^3 which moves each point less than a distance ϵ and which is the identity on the complement of a finite collection $D_1, D_2, \cdots, D_n$ of disjoint disks in $S-F$, then Property $(*, F, \text{Int } h(S))$ is also satisfied.*

PROOF. Let $p \in \text{Int } S$, $q \in \text{Ext } S$, and let ϵ be a positive number less than the distance between S and $\{p, q\}$. Suppose h is a homeomorphism and $D_1, D_2, \cdots, D_n$ is a finite collection of disks as in the statement of Lemma 1. We denote $h(D_i)$ by E_i for each i, and we let $h(S)=S'$. From the restriction on ϵ we see that $p \in \text{Int } S'$ and $q \in \text{Ext } S'$.

Let x be a point in F, and let N be an open set containing x such that $N \cap (\bigcup (E_i \cup D_i)) = \emptyset$. It follows from Theorem 11 of [7] that Property $(*, F, \text{Int } S)$ implies there is an open set U containing x such that each simple closed curve in $U \cap \text{Int } S$ can be shrunk to a point in $N-F$. With no loss in generality we assume that U is the interior of a 2-sphere. Let J be a simple closed curve in $U \cap \text{Int } S'$. We will show that $J \subset \text{Int } S$, so that J can be shrunk to a point in $N-F$. The compactness of F will then insure the following uniform property:

Property $(A, F, \text{Int } S')$: For each $\alpha > 0$ there is a $\delta > 0$ such that each simple closed curve of diameter less than δ which lies in Int S' can be shrunk to a point in an α-subset of the complement of F.

An argument similar to the proof of Theorem 2 of [7] shows that we could have begun by assuming that the set of diameters of the components of F has a positive lower bound. Then the proof of Theorem 10 of [7] insures that Property $(*, F, \text{Int } S')$ is satisfied.

All that remains is to show that $J \subset \text{Int } S$. We suppose J contains a point t in Ext S; and we let W be an arc in U from t to some point y in S such that $W \cap (S' \cup S) = \{y\}$. It follows from Theorem 14 of [2] that there is an arc X from a point r in $S - \bigcup D_i$ to p such that $X - \{r\}$ does not intersect $S \cup (\bigcup E_i)$. This means that $X - \{r\} \subset \text{Int } S \cap \text{Int } S'$. If $r \neq y$ we use the fact that Int S is 0-ulc to construct an arc Y from a point r' of X to the point y such that $Y - \{y\} \subset \text{Int } S \cap \text{Int } S'$ and $Y \cap X = \{r'\}$. Let X' be the subarc of X from p to r'. Now the arc $A = X' \cup Y \cup W$ pierces S at y. This means that there is a disk D in $S \cap U$ such that A pierces D. Since $D \subset S'$, A also pierces S' at y. This means that A intersects each complementary domain of S', yet $A - \{y\} \subset \text{Int } S'$. A similar argument (where X' is replaced by X) yields the same contradiction in the case where $r = y$. This completes the proof.

If F is a closed subset of a 2-sphere S in S^3 and V is a component of S^3-S, we define *Property* (C, F, V) to mean that for each $\epsilon>0$ there is a $\delta>0$ such that each unknotted simple closed curve that has diameter less than δ and that lies in V can be shrunk to a point in an ϵ-subset of S^3-F. Notice that Property (A, F, V), as defined in the proof of Lemma 1, implies (C, F, V). The proof of Theorem 10 of [7] shows that (A, F, V) implies Property $(*, F, V)$ if the diameters of the components of F have a positive lower bound. Actually the proof of Theorem 10 of [7] shows that (C, F, V) implies $(*, F, V)$ under the same restriction on the components of F, since the only simple closed curves that needed to be shrunk to a point were on polyhedral 2-spheres. Theorem 2 is a consequence of Theorem 1 and these observations.

THEOREM 2. *If F is a closed subset of a 2-sphere S in S^3 such that the diameters of the components of F have a positive lower bound and Property (C, F, V) is satisfied for some component V of S^3-S, then F lies on the boundary of a 3-cell.*

THEOREM 3. *Suppose F is a closed subset of a 2-sphere S in S^3 such that the diameters of the components of F have a positive lower bound. Each of the following is sufficient for F to lie on the boundary of a 3-cell:*

(a) *F can be locally spanned from V; that is, for each $\epsilon>0$ and for each point $p\in F$ there are disks R and D such that $p\in \text{Int } R\subset S$, $\text{Bd } D=\text{Bd } R$, $D-\text{Bd } D\subset V$, and $\text{diam}(D\cup R)<\epsilon$.*

(b) *F can be locally spanned in V on tame simple closed curves; that is, for each $\epsilon>0$ and for each $p\in F$ there is an ϵ-disk R such that $p\in\text{Int } R \subset S$, $\text{Bd } R$ is tame, and for each $\alpha>0$ there is an ϵ-disk D in V so that $\text{Bd } R$ can be shrunk to a point in $D\cup$(an α-neighborhood of $\text{Bd } R$).*

(c) *For each $p\in F$ and for each $\epsilon>0$ there is a 2-sphere S' of diameter less than ϵ such that $p\in\text{Int } S'$ and $S\cap S'$ is a continuum satisfying Property $(*, S\cap S', V)$.*

The proofs given for Theorems 8 and 11 of [8] need only slight modification to show that (b) and (c) each imply Property $(*, F, V)$. The proof of Theorem 7 of [3] can be adjusted as suggested in [8, p. 364] to show that (a) also implies Property $(*, F, V)$. Thus Theorem 3 also follows from Theorem 1.

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