ON FINITE GROUPS WHOSE p-SYLOW SUBGROUP IS A T. I. SET[1]

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Throughout this note we let p be a fixed prime and let G be a finite group whose fixed p-Sylow subgroup P is a T. I. set (trivial intersection set). That is, the intersection of any two distinct conjugates of P is $\langle 1\rangle$. Denote $|P|$ by p^a. It is conjectured that if G has a faithful complex character χ with $\chi(1) \leqq p^{a/2}-1$, then $P \triangle G$. This has been confirmed in certain cases [**4**, page 287 and Lemma 4.2], [**6**, Theorem 4.3]. In fact under certain conditions it is sufficient to assume $\chi(1) < (p^a-1)/2$ [**1**, Theorem 3], [**6**, Theorem 4.2], but in general the conclusion $P \triangle G$ does not hold under this weaker assumption because of the presence of Suzuki's simple groups.

Our purpose here is to use Brauer's theory of the correspondence between p-blocks of a subgroup of G and p-blocks of G [**2**], [**3**], together with a result of Gorenstein and Walter [**5**, Equation (46)], to obtain the theorem below which verifies the conjecture in the case that $C(V) \subseteq N(P)$, where V is the group of p-regular elements of $C(P)$. In particular for any counterexample of minimal order of the conjecture, we would have $C(P) \subseteq PZ(G)$.

The notation is standard. If H is a subgroup of G, then $N(H)$, $C(H)$, and $Z(H)$ denote the normalizer, centralizer, and center of H. Denote $Z(G)$ by Z. All characters are over the complex field.

Assuming P is a T. I. set, let B be a p-block of G of defect $\neq 0$, and let D be a defect group of B with $D \subseteq P$ and with $|D| = p^d$. Then $N(D) \subseteq N(P)$ and the p-Sylow group of $N(D)$ is normal in $N(D)$. Furthermore by [**2**, Theorem (8C)], there is a block $\tilde{B}$ of $N(D)$ which corresponds to B in the sense of [**2**, Theorem (7E)]. The defect group of $\tilde{B}$ is the p-Sylow group of $N(D)$ [**2**, Theorem (9F)] and is contained in D [**2**, Theorem (8D)]. We must have $D = P$. Thus every p-block of G has defect 0 or full defect a.

We know that

$$PC(P) = P \times V \tag{1}$$

where V is a group of order prime to p. Then every p-block of $PC(P)$ consists of the p^a irreducible characters $\lambda\theta$ where θ is a fixed irreducible

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character of V and λ ranges over all the irreducible characters of P. We shall denote this block by $b(\theta)$.

There is a one-to-one correspondence between the p-blocks of defect a of G and the classes $\{\theta\}$ of irreducible characters of V associated in $N(P)$ [**2**, Theorem (12A)]. Denote the block of G corresponding to $\{\theta\}$ by $B(\theta)$. Then, according to [**3**, Theorem (2D)],

$$b(\theta)^G = B(\theta) \tag{2}$$

in the sense defined there. Every p-block of $N(P)$ is of defect a and must be of the form $b(\theta)^{N(P)}$ for some θ. We denote this block by $\tilde{B}(\theta)$. Then [**3**, Theorem (2C)] implies

$$\tilde{B}(\theta)^G = B(\theta). \tag{3}$$

LEMMA 1. *An irreducible character ψ of $N(P)$ belongs to $\tilde{B}(\theta)$ if and only if $\psi | V$ has θ as a constituent.*

PROOF. Let Ω be an algebraic number field of finite degree containing the $|N(P)|$th roots of unity. Let $\mathfrak{o}$ be the ring of algebraic integers in Ω and let $\mathfrak{p}$ be a prime ideal of $\mathfrak{o}$ containing p.

If we apply (2) to $N(P)$, it follows from [2, Equation (12.2)] that for $\psi \in \tilde{B}(\theta)$ and $v \in V$ we have

$$\frac{|N(P)|\,\theta(1)}{|C(v) \cap N(P)|}\,\frac{\psi(v)}{\psi(1)} \equiv \sum_{w} \theta(w) \pmod{\mathfrak{p}}.$$

Here w ranges over the elements of V which are conjugate to v in $N(P)$. Hence

$$\frac{|N(P)|\,\theta(1)}{|C(v) \cap N(P)|}\,\frac{\psi(v)}{\psi(1)} \equiv \frac{|N(P)|}{|C(v) \cap N(P)|}\,\frac{1}{q} \sum_{\{\theta\}} \theta_i(v) \pmod{\mathfrak{p}},$$

where θ_i ranges over the associates of θ in $N(P)$ and q is the number of these associates. But, since $V \triangle N(P)$,

$$\psi | V = \frac{\psi(1)}{q'\theta'(1)} \sum_{\{\theta'\}} \theta'_j \tag{4}$$

for some class $\{\theta'\}$ where q' is the number of members of this class. These last two relations yield a congruence relating the values of θ and its associates to those of θ' and its associates. However, the irreducible characters of V are linearly independent (mod $\mathfrak{p}$) [2, Theorem (3C)]. Therefore θ and θ' are associates in $N(P)$, and the lemma follows from (4).

Let D denote the set of p-singular elements of G whose p-factor is in the fixed p-Sylow subgroup P. Let B be a p-block of G, and let $\chi_i \in B$. Then

$$\chi_i \mid N(P) = \sum_j a_{ij}\psi_j \tag{5}$$

where the ψ_j are the irreducible characters of $N(P)$ and the a_{ij} are integers. Then according to [5, Equation (46)],

$$\chi_i \mid D = \sum{}' a_{ij}\psi_j \mid D \tag{6}$$

where we have summed only those terms for which $\psi_j \in \tilde{B}$ and $\tilde{B}^G = B$ for some block $\tilde{B}$ of $N(P)$.

LEMMA 2. *If $\chi_i \in B(\theta)$ and $\chi_i(1) \leqq p^a$, then every constituent of $\chi_i \mid V$ is an associate in $N(P)$ of θ. In particular, if $\theta = 1$, then the kernel of χ_i contains V.*

PROOF. For χ_i we have an equation of the form (5). It follows from (3) and (6) that

$$\psi = \sum_{\psi_j \notin \tilde{B}(\theta)} a_{ij}\psi_j$$

vanishes on $P - \{1\}$. Hence $\psi \mid P$ must be a multiple of the character of the regular representation of P so $p^a \mid \psi(1)$. Since χ_i is not of defect 0, $\chi_i(1) < p^a$. Hence ψ is identically zero, and (5) and (6) have the same terms. The lemma now follows from Lemma 1.

In particular, $B(1_V)$ is the principal block (containing the principal character 1_G of G).

PROPOSITION. *Suppose the p-Sylow subgroup P is a T. I. set. If G has an irreducible character χ such that $\chi \mid V$ is reducible and $\chi(1) \leqq (p^a+1)^{1/2}$, then G has a normal subgroup $M \neq G$ containing V.*

PROOF. It follows from Lemma 2 that $\chi\bar{\chi}$ has a nonprincipal constituent in $B(1)$ and that this constituent has V in its kernel.

REMARK. If G has a nonprincipal character χ such that $\chi \mid V$ is irreducible then without use of the lemmas we see easily that G has a normal subgroup $M \neq G$ containing either P or V.

THEOREM. *Suppose the p-Sylow subgroup P of G is a T. I. set and that $C(V) \subseteq N(P)$. If G has a faithful character χ all of whose constituents have degrees $\leqq (p^a+1)^{1/2}$, then $P \triangle G$.*

PROOF. Suppose the statement is false and that G is a counterexample of minimal order. If for every constituent χ_0 of χ, $\chi_0 | PV$ is irreducible then $Z(P) \subseteq Z(G)$ and $P \triangle G$, which is not the case. Hence for some constituent χ_0 of χ, $\chi_0 | PV$ is reducible. Then $\chi_0\bar{\chi}_0$ has a constituent $\chi_1 \neq 1$ such that $1_{PV} \subseteq \chi_1 | PV$. By Lemma 2, $V \subseteq K$, the kernel of χ_1. Either $KN(P) = G$ or $P \triangle KN(P)$. In the first case, $\chi_1 | N(P)$ is irreducible and then $P \subseteq K \triangle G$. By the minimality of G, $P \triangle K \triangle G$, which is not the case.

Thus $P \triangle KN(P)$. Then $K \cap P = 1$ since $P \not\triangle G$. Hence $KP = K \times P$, so $V \subseteq K \subseteq V$ and $V \triangle G$. Then $P \triangle VC(V) \triangle G$ so $P \triangle G$. This is a contradiction and the proof is complete.

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