ON A CONJECTURE OF DEMARR ON PARTIALLY ORDERING TOPOLOGICAL SPACES

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1. **Introduction.** Let X be a topological space. We will say that X has *property* (L) if X can be partially ordered so that it becomes a complete lattice in which order-convergence [**1**, p. 59] of arbitrary nets coincides with the topological convergence. In [**2**] Ralph DeMarr has raised the question as to whether every compact Hausdorff space possessed property (L). That such is not the case will be shown below. In fact, there is a familiar class of topological spaces each infinite, compact member of which fails to have the property: namely, the so-called F-spaces of Gillman and Jerison [**3**]. An F-space is Hausdorff by definition.

2. **Preliminaries.** Although we will show that not every compact Hausdorff space has property (L), it remains an interesting (and open) question as to what general classes of topological spaces, if any, do possess the property. Also, one may wish to consider relaxing the conditions of property (L) without, of course, destroying entirely the potential utility of the order structure in obtaining further insight into the topological structure of the space. With this in mind, we introduce the following definitions due to McShane [**4**].

DEFINITION 1. If $(D, \geqq)$ is a directed set and ξ is a net from D to the partially ordered set $(X, \leqq)$, we say that ξ o-converges to $x \in X$ iff there exists a pair of nonempty subsets M and N of X such that

(i) M is up-directed, N is down-directed;

(ii) $x = \bigvee M = \bigwedge N$;

(iii) for each $m \in M$ and $n \in N$, there exists $\beta \in D$ such that $m \leqq \xi_\alpha \leqq n$ for all $\alpha \geqq \beta$ in D.

DEFINITION 2. A partially ordered set X is said to be *Dedekind complete* iff for every nonempty subset M of X which is up-directed and has an upper bound in X, the supremum $\bigvee M$ exists in X (and dually for nonempty down-directed subsets).

A topological space X will be said to have *property* (P) if X can be endowed with a binary relation $\leqq$ so that it becomes a Dedekind complete partially ordered set in which o-convergence coincides with topological convergence.

REMARK. It is clear that property (P) is, in general, weaker than

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property (L). In fact, for arbitrary complete lattices the two notions of convergence and completeness coincide.

3. **Compact Hausdorff spaces which do not have property** (P). We shall prove that no infinite compact F-space has property (P). We do this by showing that if X is any infinite compact topological space having property (P), then X contains a convergent sequence of distinct points. However, if X is an F-space, it is a known fact that no point of the space can be the limit of a sequence of distinct points in X (cf. [**3**, Example 14N, pp. 208–218]). We require two lemmas, the first of which is known [**4**, p. 15].

Lemma 1. *If X is a partially ordered set and $\xi=\{\xi_n\colon n\in\omega\}$ is a monotone increasing (decreasing) sequence in X such that $\bigvee\xi(\omega)$ $(\bigwedge\xi(\omega))$ exists, then ξ o-converges to $\bigvee\xi(\omega)$ $(\bigwedge\xi(\omega))$.*

Lemma 2. *Let X be an infinite partially ordered set in which arbitrary sequences possess o-convergent subnets. Then X contains the range of at least one bounded monotone sequence of distinct elements.*

Proof. Since X is infinite, X contains a sequence $\xi=\{\xi_n\colon n\in\omega\}$ of distinct elements, which, by hypothesis, possesses a subnet $\eta=\{\xi_{\phi(\ )}\colon \alpha\in D\}$ o-converging to some element, x say, in X. Maintaining the notation in Definition 1, let M and N satisfy conditions (i)–(iii) for η and x. We assert that $x\notin M\cap N$. For suppose $x\in M\cap N$. Then by condition (iii), there exists $\beta\in D$ such that $x\leqq\xi_{\phi(\alpha)}\leqq x$ for all $\alpha\geqq\beta$. Also, by the definition of subnet, given $n>\phi(\beta)$ in ω there exists $\gamma\in D$ such that $\alpha\geqq\gamma$ implies $\phi(\alpha)\geqq n$. Hence, choosing $\delta\geqq$ both β and γ, we have that $\xi_{\phi(\delta)}=\xi_{\phi(\beta)}=x$ and $\phi(\delta)\geqq n>\phi(\beta)$. But this contradicts the assumption that the image points of ξ are distinct.

Thus we may assume that $x\notin M\cap N$. Suppose $x\notin M$ and select $m_1\in M$ arbitrarily. We shall construct inductively the desired sequence. Since $m_1<x$, m_1 is not an upper bound for M and so there exists m_1' in M such that $m_1\ngeqq m_1'$. Since M is up-directed, there exists $m_2\in M$ such that $m_2\geqq$ both m_1 and m_1'. Surely $m_1\neq m_2$, and so we must have $m_1<m_2$. Since $m_2<x$, the same argument shows there exists $m_3\in M$ such that $m_1<m_2<m_3<x$. Continuing in the above manner, we obtain a bounded monotone sequence of distinct elements, thus completing the proof in this case. The case when $x\notin N$ follows dually.

Theorem. *There exist compact Hausdorff spaces which do not possess property* (P).

Proof. Let (X,τ) be any infinite compact F-space, and suppose that X has property (P). By compactness, every net in X has a τ-convergent subnet. Thus, by property (P), every net has an o-convergent subnet, and X satisfies the hypothesis of Lemma 2. Accordingly, let $\xi = \{\xi_n : n \in \omega\}$ be a bounded monotone sequence of distinct elements of X. We suppose ξ is increasing (the case for ξ decreasing is entirely dual). Since X is Dedekind complete, $\bigvee \xi(\omega)$ exists in X and, by Lemma 1, ξ o-converges to $\bigvee \xi(\omega)$. Therefore ξ τ-converges to $\bigvee \xi(\omega)$. But this is inconsistent with the assumption that X is an F-space, thereby completing the proof.

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