ON PRINCIPAL SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

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1. **Introduction.** In this paper we investigate the asymptotic behavior of solutions of certain nth order nonhomogeneous linear ordinary differential equations, $\Omega(y)=\phi$, near a singular point at ∞. The class of nth order linear differential operators, Ω, treated here consists roughly of those whose coefficients are complex functions, defined and analytic in unbounded sectorial regions, and have asymptotic expansions as $x\to\infty$ in terms of real (but not necessarily integral) powers of x and/or functions (called trivial) which are of smaller rate of growth ($\prec$) than all powers of x as $x\to\infty$. (We are using here the concept of asymptotic equivalence ($\sim$) as $x\to\infty$ and the order relations "$\prec$" introduced in [**3**, §13]. However, it should be noted (see [**3**, §128(g)]) that the class of operators treated here includes as a special case those operators where no requirement is imposed except that each coefficient be analytic and have an asymptotic expansion (in the customary sense) of the form $\sum C_j x^{-\lambda_j}$ with λ_j real and $\lambda_j\to+\infty$ as $j\to\infty$. (A summary of the necessary definitions from [**3**] appears in §2 below.)

In [**5**], Strodt showed that if ϕ is a nontrivial analytic function which also possesses, as $x\to\infty$, an asymptotic expansion in terms of real powers of x and/or trivial functions, then the equation $\Omega(y)=\phi$ has at least one solution y_0 which is $\sim$ to a logarithmic monomial (i.e., a function of the form $Kx^{\alpha_0}(\log x)^{\alpha_1}(\log\log x)^{\alpha_2}\cdots(\log_q x)^{\alpha_q}$ for complex $K\neq 0$ and real α_j) and such that if $f\prec y_0$, then $\Omega(f)\prec\phi$. (A solution with these two properties is called a *principal solution* in [**3**, §69] and is clearly of minimal rate of growth at ∞.)

In this paper we consider the case where ϕ is *any* function $\sim$ to a logarithmic monomial, and we show in §3 that the equation $\Omega(y)=\phi$ always has a principal solution. As a corollary (§5) we apply a result proved in [**1**, §12] to obtain a representation theorem for those solutions of $\Omega(y)=\phi$ which are $\prec$ some power of x. Our method of proving §3 consists of first obtaining a sufficiently close approximate solution by successive integrations of a factored equation, and then using the approximate solution to transform the equation into one in which an exact solution can be obtained using [**4**, §99].

2. **Concepts from** [**3**]. (a) [**3**, §94]. Let $-\pi\leqq a\leqq b\leqq\pi$. For each non-

Received by the editors March 2, 1967.

negative real-valued function g on $(0, (b-a)/2)$, let $V(g)$ be the union (over $\delta \in (0, (b-a)/2)$) of all sectors $a+\delta < \arg(x-h(\delta)) < b-\delta$ where $h(\delta)=g(\delta)\exp(i(a+b)/2)$. The set of all $V(g)$ (for all choices of g) is denoted $F(a, b)$ and is a filter base which converges to ∞. A statement is said to hold *except in finitely many directions* (briefly, *e.f.d.*) in $F(a, b)$ if there are finitely many points $r_1 < \cdots < r_q$ in (a, b) such that the statement holds in each of $F(a, r_1)$, $F(r_1, r_2)$, $\cdots$, $F(r_q, b)$ separately.

(b) [**3**, §13]. If f is analytic in some $V(g)$, then $f \to 0$ in $F(a, b)$ means that for any $\epsilon > 0$, there is a g_1 such that $|f(x)| < \epsilon$ for all $x \in V(g_1)$. $f \prec 1$ means that in addition to $f \to 0$, all functions $\theta_j^k f \to 0$ where $\theta_j f = x \log x \cdots \log_{j-1} x f'$ and where θ_j^k is the kth iteration of the operator θ_j. Then $f_1 \prec f_2$, $f_1 \sim f_2$, $f_1 \approx f_2$ mean respectively $f_1/f_2 \prec 1$, $f_1 - f_2 \prec f_2$ and $f_1 \sim c f_2$ for some constant $c \neq 0$. If $f \sim c$, we write $f(\infty) = c$, while if $f \prec 1$, we write $f(\infty) = 0$. If $M = x^{\alpha_0}(\log x)^{\alpha_1} \cdots (\log_r x)^{\alpha_r}$ for some r and M is not constant, then by [**3**, §28] $f \prec M$ implies $f' \prec M'$. If $f \approx M$, then $\delta_k(f)$ will denote α_k. If $j \geqq i$, then $s_{ji}(\alpha)$ will denote the elementary symmetric function of degree i in $\alpha, \alpha-1, \cdots, \alpha-j+1$.

(c) [**3**, §49]. A logarithmic domain of rank zero (briefly an LD_0) over $F(a, b)$ is a complex vector space E of functions (each analytic in some $V(g)$) which contains the constants and such that any finite linear combination of elements of E, with coefficients which for some $q \geqq 0$ are functions of the form $cx^{\alpha_0}(\log x)^{\alpha_1} \cdots (\log_q x)^{\alpha_q}$ (for real α_j), is either $\sim$ to a function of this latter form or is trivial.

3. **The main theorem.** *Consider the equation $\Omega(y) = \phi$, where $\Omega(y)$ is an nth order linear differential polynomial with coefficients in an LD_0 over $F(a, b)$, and where ϕ is a function which in $F(a, b)$ is $\sim$ to a logarithmic monomial. If θ is the operator $\theta y = xy'$, $\Omega(y)$ may be written $\Omega(y) = \sum_{i=0}^{n} B_i(x)\theta^i y$, where the functions B_j belong to an LD_0. We assume B_n is nontrivial. By dividing the equation $\Omega(y) = \phi$ through by the highest power of x which is $\sim$ to a coefficient B_j, we may assume that for some $m \geqq 0$, $B_m \approx 1$ while for each j, $B_j \prec 1$ or $B_j \approx 1$. Let $F(\alpha) = \sum_{i=0}^{n} B_i(\infty)\alpha^i$. Let Q be the logarithmic monomial such that $\phi \sim Q$ and let $\delta_j(Q) = \sigma_j$ for each j. Define a logarithmic monomial M as follows. If $F(\sigma_0) \neq 0$, let $M = (F(\sigma_0))^{-1}Q$. If σ_0 is a root of F of multiplicity r, then let*

$$M = (F^{(r)}(\sigma_0)/r!)^{-1}(s_{rr}(\sigma_1 + r))^{-1}(\log x)^r Q \qquad \text{if} \quad \sigma_1 \notin \{-1, -2, \cdots, -r\},$$

while if $\sigma_1 \in \{-1, -2, \cdots, -r\}$, let

$$M = c(\log x)^r(\log_2 x \cdots \log_k x)Q$$

where $k = \min\{j: j \geqq 2, \sigma_j \neq -1\}$, *and*

$$c = (s_{r,r-1}(\sigma_1 + r))^{-1}(\sigma_k + 1)^{-1}(F^{(r)}(\sigma_0)/r!)^{-1}.$$

Then: (1) *The equation* $\Omega(y) = \phi$ *possesses at least one solution* $y_0 \sim M$ *e.f.d. in* $F(a, b)$. (2) *If* y_0 *is a solution of* $\Omega(y) = \phi$ *such that* $y_0 \sim M$ *in some* $F(a_1, b_1)$, *then for any function* f *which is* $\prec y_0$ *in* $F(a_1, b_1)$, *we have* $\Omega(f) \prec \phi$ *in* $F(a_1, b_1)$. *In particular, among all solutions of* $\Omega(y) = \phi$ *in* $F(a_1, b_1)$, y_0 *is of minimal rate of growth at* ∞.

PROOF OF PART (2). We consider $\Omega(y) - Q$ and apply the algorithm introduced in [**3**, §66] which produces the set of those logarithmic monomials N (called principal monomials in [**3**, §67]) such that $\Omega(N) \sim Q$ and $\Omega(f) \prec Q$ whenever $f \prec N$. For $\Omega(y) - Q$ we find by applying the algorithm that M is the unique principal monomial. Hence if $f \prec M$, then $\Omega(f) \prec Q$. Since $y_0 \sim M$ and $\phi \sim Q$, part (2) clearly follows.

The proof of part (1) will be based on a sequence of lemmas and will be concluded in §6.

4. **Lemma.** *Let* γ *be a complex number and let* ψ *be a function which in* $F(a, b)$ *is* $\sim$ *to a logarithmic monomial* R. *Let* $\delta_i(R) = \lambda_i$. *Define a logarithmic monomial* N *as follows*:

(a) *If* $\lambda_0 \neq \gamma$, *let* $N = (\lambda_0 - \gamma)^{-1}R$.

(b) *If* $\lambda_0 = \gamma$, *let* $N = (\lambda_q + 1)^{-1}(\log x \cdots \log_q x)R$ *where* $q = \min\{j: j \geqq 1, \lambda_j \neq -1\}$.

Then in $F(a, b)$, *the equation* $xy' - \gamma y = \psi$ *has at least one solution* $y^* \sim N$.

PROOF. Under the change of variable $y = x^{\lambda_0}z$ and multiplication by $x^{-\lambda_0}$, the equation $xy' - \gamma y = \psi$ becomes

$$xz' + (\lambda_0 - \gamma)z = \psi_0 \tag{1}$$

where $\psi_0 = x^{-\lambda_0}\psi$.

Let $N_0 = x^{-\lambda_0}N$. The proof is divided into three cases.

Case A. $\operatorname{Re}(\lambda_0 - \gamma) \neq 0$.

In this case, under $z = N_0 + N_0 w$ and division by $(\lambda_0 - \gamma)N_0$, equation (1) becomes,

$$x(\lambda_0 - \gamma)^{-1}w' + f(x)w = g(x), \tag{2}$$

where $f \sim 1$ (since $xN_0' \prec N_0$ by a simple calculation) and where $g \prec 1$ since $(\lambda_0 - \gamma)N \sim \psi$. Thus (2) is normal in the sense of [**3**, §83] with divergence monomial $(\lambda_0 - \gamma)x^{-1}$. Since $d = \operatorname{Re}(\lambda_0 - \gamma) \neq 0$, it follows from [3, §111] (when $d>0$) and [3, §117] (when $d<0$) that (2) possesses a solution $w_0 \prec 1$ in $F(a, b)$. Then clearly $y^* = x^{\lambda_0}(N_0 + N_0 w_0)$ is $\sim N$ and satisfies the equation $xy' - \gamma y = \psi$.

Case B. $\lambda_0 = \gamma$.

Thus (1) is of the form $z' = x^{-1}\psi_0$. With N as defined in (b) above, it is proved in [2, p. 272] that for some constant A, $z_0 = A + \int_{x_0}^{x} x^{-1}\psi_0$ is $\sim N_0$ in $F(a, b)$. Hence if $y^* = x^{\lambda_0} z_0$, then y^* satisfies the conclusion.

Case C. $\mathrm{Re}(\lambda_0 - \gamma) = 0$ and $\lambda_0 \neq \gamma$.

In this case, (1) may be written $xz' - \sigma i z = \psi_0$ where $\sigma = i(\lambda_0 - \gamma)$ is a nonzero real number. Under $z = -(\sigma i)^{-1}\psi_0 + w$, this becomes $xw' - \sigma i w = \psi_1$ where $\psi_1 = (\sigma i)^{-1} x \psi_0'$. Since $\psi_0 \prec (\log x)^{\lambda_1 + 1/2}$, $\psi_1 \prec (\log x)^{\lambda_1 - 1/2}$ by §2(b), and hence $\psi_1 \prec \psi_0$ since $(\log x)^{\lambda_1 - \epsilon} \prec \psi_0$ for all $\epsilon > 0$. Under $w = -(\sigma i)^{-1}\psi_1 + u$, we obtain $xu' - \sigma i u = \psi_2$ where by §2(b), $\psi_2 \prec (\log x)^{\lambda_1 - 3/2}$ (thus $\psi_2 \prec \psi_0$ since $\psi_0 \succ (\log x)^{\lambda_1 - \epsilon}$ for all $\epsilon > 0$). Clearly this process can be repeated so as to make the constant term $\prec (\log x)^{\alpha}$ for α as small as desired. Hence there is a function $f \sim -(\sigma i)^{-1}\psi_0$ in $F(a, b)$ such that under $z = f + v$, equation (1) becomes

$$xv' - \sigma i v = \phi_1 \tag{3}$$

where ϕ_1 is chosen so that,

$$\phi_1 \prec (\log x)^{-1-t} \tag{4}$$

where $t = 1 + \max\{0, -2\lambda_1\}$.

The technique we now employ to prove the existence of a solution $v_0 \prec f$ of (3) is similar to the technique used by Strodt in the proof of [6, Section 107].

Let $E_1 \in F(a, b)$ be such that on E_1, we have $|x| \geqq 2$ and

$$|\phi_1(x)| \leqq (\log |x|)^{-1-t/2}. \tag{5}$$

For x and x_1 in E_1, let $B(x, x_1) = \exp \int_x^{x_1} (-\sigma i/u)\, du$, where the contour is any rectifiable path from x to x_1 in E_1. Then clearly, if we put $L(x, \rho) = B(x, \rho x)$ for $1 \leqq \rho < \infty$, we have

$$|L(x, \rho)| \equiv 1 \quad \text{and} \quad \partial L(x, \rho)/\partial x \equiv 0. \tag{6}$$

Hence,

$$|L(x, \rho)|\, \rho^{-1} |\phi_1(\rho x)| \leqq (\rho \log 2\rho)^{-1} (\log 2\rho)^{-t/2}$$

for $x \in E_1$ and $1 \leqq \rho < \infty$; and since the right side is

$$(-2/t)\, d((\log 2\rho)^{-t/2})/d\rho,$$

we have by the M-test [7, p. 22] that the integral

(7) $$v_0(x) = -\int_1^\infty L(x, \rho)\rho^{-1}\phi_1(\rho x)d\rho$$

is uniformly convergent on E_1 and thus represents an analytic function there (e.g., [7, p. 100]) whose derivative may be calculated by differentiating under the integral sign. In view of (5), clearly

(8) $$|v_0(x)| \leqq (2/t)(\log |x|)^{-t/2}$$

on E_1 and hence $v_0 \to 0$ in $F(a, b)$. Differentiating (7), we see easily that $v_0' - \sigma i x^{-1} v_0 = x^{-1}\phi_1$ in E_1, so v_0 is a solution of (3). Successively differentiating (7) and using (6), we see that for all j

(8a) $$\theta^j v_0(x) = -\int_1^\infty L(x, \rho)\rho^{-1}(\theta^j\phi_1)(x\rho)d\rho$$

(where, for example, $(\theta\phi_1)(x\rho) = x\rho\phi_1'(x\rho)$ etc.) in E_1. Since $\phi_1 \prec (\log x)^{-1-t/2}$, it follows (see §2(b)) that $\theta^j\phi_1 \prec (\log x)^{-(j+1)-t/2}$ in $F(a, b)$, and so, for each j, there is an $S_j \in F(a, b)$ and a constant c_j such that

$$|\theta^j\phi_1(x)| \leqq c_j(\log |x|)^{-(j+1)-t/2} \text{ on } S_j.$$

Hence by (8a), there is a C_j' such that

(9) $$|\theta^j v_0(x)| \leqq C_j'(\log |x|)^{-j-t/2} \text{ in } S_j.$$

Thus $\theta^j v_0 \to 0$ in $F(a, b)$ for each j. Now let $p > 1$. Then, by the definition of the operator θ_p, $\theta_p v_0 = G\theta v_0$ where $G = \log x \cdots \log_{p-1} x$. It is routine to verify by induction on j that for $j = 1, 2, \cdots$

(10) $$\theta_p^j v_0 = \sum_{\alpha=1}^{j} G_{\alpha j}\theta^\alpha v_0,$$

where

(11) $$G_{\alpha j} = \sum m(i_1, \cdots, i_j, \alpha)G^{i_1}(\theta G)^{i_2} \cdots (\theta^{j-1}G)^{i}$$

in which the m's are constants, $i_1 + \cdots + i_j = j$ and $i_2 + 2i_3 + \cdots + (j-1)i_j = j - \alpha$ for each term in (11). Now for all $\epsilon > 0$, $G \prec (\log x)^{1+\epsilon}$ so (see §2(b)) $\theta^j G \prec (\log x)^{1-j+\epsilon}$ for each j. Hence, by (11), for each α and j we have $G_{\alpha j} \prec (\log x)^{\alpha+\epsilon j}$ for *all* $\epsilon > 0$. Now t is a fixed positive number and so for each given α and j, we have, by taking $\epsilon = t/5j$, that $G_{\alpha j} \prec (\log x)^{\alpha+t/4}$. Hence there exist $S_{\alpha j} \in F(a, b)$ and constants $d(\alpha, j)$ such that on $S_{\alpha j}$,

$$|G_{\alpha j}(x)| \leqq d(\alpha, j)(\log |x|)^{\alpha+t/4}.$$

Thus by (9) and (10), for each p and j,

$$|\theta_p{}^j v_0(x)| \leqq m_{pj}(\log |x|)^{-t/4}$$

in some element of $F(a, b)$ for some constant m_{pj}. Thus $\theta_p^j v_0 \to 0$ for all p and j since $t>0$ and so

(12) $$v_0 \prec 1 \text{ in } F(a, b).$$

Since v_0 solves (3), we have

(13) $$v_0 = (\sigma i)^{-1}(xv_0' - \phi_1).$$

Since $v_0 \prec 1$, $xv_0' \prec (\log x)^{-1}$. Thus since $\phi_1 \prec (\log x)^{-1-t/2}$, we have by (13) that $v_0 \prec (\log x)^{-1}$. Hence $xv_0' \prec (\log x)^{-2}$, and so if $-1-t/2 < -2$, we have $v_0 \prec (\log x)^{-2}$. Continuing this way, if m is the greatest integer $<1+t/2$, then $v_0 \prec (\log x)^{-m}$, and so since $m+1 \geqq 1+t/2$, we have by (13) that $v_0 \prec (\log x)^{-1-t/2}$ in $F(a, b)$. Thus by (4), $v_0 \prec (\log x)^{\lambda_1 - 1}$ and so $v_0 \prec f$ in $F(a, b)$. Hence if $z_0 = f + v_0$, then $z_0 \sim f$ and z_0 solves (1). Hence $y^* = x^{\lambda_0} z_0$ is a solution of $xy' - \gamma y = \psi$ and $y^* \sim N$ in $F(a, b)$ concluding the proof.

5. **Lemma.** *Assume the hypothesis and notation of* §3. *Let* $\Phi(y) = \sum_{i=0}^n B_i(\infty)\theta^i y$. *Then there exists a function* y^* *such that* $\Phi(y^*) = \phi$ *and* $y^* \sim M$ *in* $F(a, b)$ (*where* M *is as in* §3).

Proof. Let $F(\alpha) = \sum B_i(\infty)\alpha^i$ be of degree p. If $p=0$, take $y^* = (B_0(\infty))^{-1}\phi$. Hence we may assume $p>0$. It is easy to verify that if $F(\alpha) = b_p(\alpha-\alpha_1) \cdots (\alpha-\alpha_p)$ (where $b_p = B_p(\infty)$), then $b_p^{-1}\Phi = (\theta-\alpha_1) \circ \cdots \circ (\theta-\alpha_p)$ where the order of the factors is immaterial. Let $\phi^* = b_p^{-1}\phi$ and let $Q^* = b_p^{-1}Q$. We solve $\Phi(y) = \phi$ by successive integrations on $b_p^{-1}\Phi = \phi^*$ using §4, and we adopt the following notation. We let y_1 be any solution of $xy' - \alpha_1 y = \phi^*$ given by §4. Since y_1 is $\sim$ to a logarithmic monomial, we let y_2 be any solution of $xy' - \alpha_2 y = y_1$ given by §4. In general, y_{j+1} is any solution of $xy' - \alpha_{j+1} y = y_j$ $(1 \leqq j \leqq p-1)$ given by §4. Then clearly $y^* = y_p$ solves $\Phi(y) = \phi$. We will show $y^* \sim M$.

Case I. $F(\sigma_0) \neq 0$. By §4(a), $y_1 \sim (\sigma_0 - \alpha_1)^{-1}Q^*$. Similarly $y_2 \sim (\sigma_0-\alpha_2)^{-1}(\sigma_0-\alpha_1)^{-1}Q^*$. Continuing by §4(a), $y_p \sim (F(\sigma_0))^{-1}Q$ so $y_p \sim M$.

Case II. σ_0 is a root of F of multiplicity r and $\sigma_1 \notin \{-1, \cdots, -r\}$. Let $\alpha_1 = \cdots = \alpha_r = \sigma_0$. By §4(b), $y_1 \sim (\sigma_1+1)^{-1}(\log x)Q^*$. Similarly by §4(b),

$$y_j \sim (\sigma_1 + j)^{-1} \cdots (\sigma_1 + 1)^{-1}(\log x)^j Q^* \quad \text{for } 2 \leqq j \leqq r.$$

Then by §4(a),

$$y_{r+1} \sim (\sigma_0 - \alpha_{r+1})^{-1}(\sigma_1 + r)^{-1} \cdots (\sigma_1 + 1)^{-1}(\log x)^r Q^*$$

and by continuing to use §4(a), clearly

$$y_p \sim K(\log x)^r Q \quad \text{where } K = (F^{(r)}(\sigma_0)/r!)^{-1}(s_{rr}(\sigma_1 + r))^{-1}$$

so $y_p \sim M$.

Case III. σ_0 is a root of F of multiplicity r and $\sigma_1 = -1$. Thus, $\min\{j: j \geqq 1, \sigma_j \neq -1\} = k$ (as in §3). Thus, by §4(b), (assuming $\alpha_1 = \cdots = \alpha_r = \sigma_0$),

$$y_1 \sim (\sigma_k + 1)^{-1}(\log x \cdots \log_k x)Q^*.$$

Since $\delta_1(y_1) = 0$, we find by continuing up to r using §4(b) that $y_r \sim ((r-1)!)^{-1}(\log x)^{r-1}y_1$. We now continue using §4(a) and find that $y_p \sim b_p(F^{(r)}(\sigma_0)/r!)^{-1}y_r$. Since $(r-1)! = s_{r,r-1}(\sigma_1 + r)$, clearly $y_p \sim M$.

Case IV. σ_0 is a root of F of multiplicity r and $\sigma_1 \in \{-2, \cdots, -r\}$. Let $s = -\sigma_1$. Since $\sigma_1 \neq -1$, by §4(b), (assuming $\alpha_1 = \cdots = \alpha_r = \sigma_0$), $y_1 \sim (\sigma_1 + 1)^{-1}(\log x)Q^*$. Continuing up to $s-1$, we find

$$y_{s-1} \sim [(\sigma_1 + 1) \cdots (\sigma_1 + s - 1)]^{-1}(\log x)^{s-1}Q^*.$$

Since $\delta_1(y_{s-1}) = -1$, we have by §4(b),

$$y_s \sim (\sigma_k + 1)^{-1} \log x \cdots \log_k x y_{s-1}.$$

Since $\delta_1(y_s) = 0$, we have, using §4(b), that $y_r \sim ((r-s)!)^{-1}(\log x)^{r-s}y_s$. Now, using §4(a), we find $y_p \sim b_p(F^{(r)}(\sigma_0)/r!)^{-1}y_r$. Since $(r-s)! = (\sigma_1 + r) \cdots (\sigma_1 + s + 1)$, it follows that $y_p \sim M$.

6. **Conclusion of main theorem** (§3). For each i, $B_i = b_i + w_i$ where $b_i = B_i(\infty)$ and $\delta_0(w_i) < 0$. Letting $\Phi(y) = \sum_{i=0}^n b_i\theta^i y$ and $\Gamma(y) = \sum_{i=0}^n w_i\theta^i y$, we have by §5 that there exists a function $y^* \sim M$ in $F(a, b)$ such that $\Phi(y^*) = \phi$. Under $y = y^* + z$, $\Omega(y) = \phi$ becomes $\Omega(z) = -\Gamma(y^*)$. Now if $\delta_0(y^*) = \lambda$, then it is easily verified that $\delta_0(\theta^j y^*) \leqq \lambda$ for each j. Letting $\epsilon > 0$ be such that $\delta_0(w_i) < -\epsilon$ for each i, we have

$$(1) \qquad \delta_0(\Gamma(y^*)) < \lambda - \epsilon.$$

We now utilize a technique employed by Strodt in [5] which we outline here for the reader's convenience. Let $H = \{\alpha: F(\alpha) = 0\}$. Then if q is a real number not in H and we let $k_q = (F(q))^{-1}$, it is easily seen that the principal monomial of $\Omega(y) - x^q$ is $k_q x^q$. Hence if we let $\Lambda_q(\omega) = x^{-q}\Omega(k_q x^q \omega)$, then by the properties of a principal monomial we have $\Lambda_q(1) \sim 1$ and $\Lambda_q(E) \prec 1$ if $E \prec 1$. (Thus Λ_q is unimajoral in the terminology of [4, Section 13]). Further, it is easily seen that Λ_q has coefficients in an $LD_0(F(a, b))$ and that $\partial\Lambda_q/\partial\omega^{(n)}$ is a nontrivial function. Thus by [4, Section 27], Λ_q possesses at least one principal factorization sequence, that is, a sequence $(V_1, \cdots, V_n)$ of logarithmic monomials such that Λ_q may be written

$$\Lambda_q = \dot{V}_n \cdots \dot{V}_1 + \sum_{j=0}^{n} E_j \dot{V}_j \cdots \dot{V}_1$$

where $\dot{V}_j$ is the operator $\dot{V}_j(y) = y - y'/V_j$ and where each $E_j \prec 1$. Now by definition of Λ_q, it is easily verified that

$$\Lambda_q(\omega) = k_q \sum_{j=0}^{n} B_j(q+\theta)^j \omega,$$

and so it follows from [**4**, Section 44] that all principal factorization sequences for $\Lambda_q(\omega)$ can be obtained as follows. If we let

$$C_1^* \Lambda_q(y) = k_q \sum_{j=0}^{n} B_j(q + xy)^j$$

and if $N_1, \cdots, N_n$ are the logarithmic monomials such that the zeros $y_1, \cdots, y_n$ of $C_1^*\Lambda_q(y)$ satisfy $y_j/N_j \to 1$ for each j, then $(V_1, \cdots, V_n)$ is a principal factorization sequence for Λ_q if and only if $(V_1, \cdots, V_n)$ is a permutation of $(N_1, \cdots, N_n)$ and for each j, V_j is either $\prec$ or $\approx$ to V_{j+1}. Since $\{B_0, \cdots, B_n\}$ is contained in an $LD_0(F(a, b))$, it easily follows that if $(V_1, \cdots, V_n)$ is a principal factorization sequence for Λ_q, then for each j, V_j has the form

$$V_j = c_j x^{-1+t_j} \tag{2}$$

for some constant c_j and some $t_j \geqq 0$. V_j is called nonexceptional if either $t_j > 0$ or c_j is not purely imaginary, and $(V_1, \cdots, V_n)$ is called nonexceptional if each V_j is nonexceptional. From the definition of $C_1^*\Lambda_q(y)$, it follows that for q and s not in H, we have

$$C_1^*\Lambda_q(y) = k_q(k_s)^{-1} C_1^*\Lambda_s((q-s)x^{-1} + y).$$

We now fix s and we fix a principal factorization sequence for Λ_s (which may be exceptional). By the above relation, $y^\#$ is a zero of $C_1^*\Lambda_q(y)$ if and only if $(q-s)x^{-1}+y^\#$ is a zero of $C_1^*\Lambda_s(y)$, and so it easily follows from the previous discussion that except for finitely many real q, $\Lambda_q(\omega)$ possesses a nonexceptional principal factorization sequence. Thus we have outlined the proof given in [**5**] that there is a finite set G of real numbers such that for any real number $q \notin G$, $\Lambda_q(\omega)$ is unimajoral and possesses a nonexceptional principal factorization sequence.

In our case here, we choose a real number $q \notin G$ such that $\lambda - \epsilon < q < \lambda$, and let $(V_1, \cdots, V_n)$ be a nonexceptional principal factorization sequence for this Λ_q. If V_j has the form in (2), then its indicial function (as defined in [4, Section 61]) is the function defined on (a, b) given by $f_j(\alpha)=\cos(t_j\alpha+\arg c_j)$. (Thus if $t_j=0, f_j$ is the constant function $\cos(\arg c_j)$.) Since each V_j is nonexceptional, each f_j has only finitely many zeros in (a, b). Hence the union of the sets of roots of the f_j is a finite set $\gamma_1 < \cdots < \gamma_d$ in (a, b). Thus if $I=(a_1, b_1)$ is any of the intervals $(a, \gamma_1), (\gamma_1, \gamma_2), \cdots, (\gamma_d, b)$, then no indicial function vanishes anywhere in I, so by definition [4, §98], $(V_1, \cdots, V_n)$ is unblocked in (a_1, a_2, b_1) for any a_2. Furthermore, by (1) and choice of q, $x^{-q}\Gamma(y^*) \prec 1$, and so by definition [4, §88], $(V_1, \cdots, V_n)$ is a strong factorization sequence for $\Lambda_q(\omega)+x^{-q}\Gamma(y^*)$ in $F(I)$. Hence by [4, §99, Theorem II], the equation $\Lambda_q(\omega)+x^{-q}\Gamma(y^*)=0$ possesses a solution $\omega_0 \prec 1$ in $F(I)$. Letting $z_0=k_q x^q \omega_0$, we have $\Omega(z_0)=-\Gamma(y^*)$; and since $q<\lambda$, we have $z_0 \prec y^*$. Letting $y_0=y^*+z_0$, we then have $\Omega(y_0)=\phi$ and $y_0 \sim M$ in $F(I)$, which concludes the proof.

7. **Corollary.** *Under the hypothesis and notation of* §3, *let I be a subinterval of (a, b) such that in $F(I)$ there is a solution $y_0 \sim M$ of $\Omega(y)=\phi$ (as just proved) and such that a complete logarithmic set of solutions $\{g_1, \cdots, g_p\}$ of $\Omega(y)=0$ exists in $F(I)$. (It was shown in* [1, §11] *that if $p=\max\{j: B_j(\infty)\neq 0\}$, then e.f.d. in $F(a, b)$ there exist solutions $g_1, \cdots, g_p$ of $\Omega(y)=0$ such that $g_j \sim x^{\alpha_j}(\log x)^{\beta_j}$ for some complex α_j and integer β_j and such that $(\alpha_k, \beta_k)\neq(\alpha_j, \beta_j)$ if $k\neq j$.)*

Then if $y^\#$ is any solution of $\Omega(y)=\phi$ which in $F(I)$ is $\prec x^\delta$ for some constant δ, then e.f.d. in $F(I)$ there exist constants $c_1, \cdots, c_p$ and a trivial function $T(x)$ such that $y^\# = y_0 + \sum_{j=1}^{p} c_i g_i + T$.

Proof. $y^\# - y_0$ is a solution of $\Omega(y)=0$ and is $\prec x^\alpha$ for some α so the result follows immediately from [1, §12].

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