AN OSCILLATION CRITERION FOR SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS

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1. **Introduction.** The well-known Leighton-Wintner [2], [3] oscillation theorem for

$$y'' + p(x)y = 0 \qquad p(x) \text{ continuous on } [0, \infty) \tag{1}$$

is

THEOREM (i). *Equation* (1) *is oscillatory on* $[0, \infty)$ *if*

$$\int^{\infty} p = +\infty. \tag{2}$$

The case

$$\liminf_{x\to\infty} \int^{x} p < +\infty \tag{3}$$

remains of interest and can produce either oscillatory or nonoscillatory behavior.

Let

$$P(x) = \frac{1}{x}\int_0^x \int_0^t p(s)\,ds\,dt. \tag{4}$$

Hartman [1] has proved that nonoscillation of (1) implies that either $P(x)$ tends to a finite limit or else that $\liminf_{x\to\infty} P(x) = -\infty$, so that one has:

THEOREM (ii). $-\infty < \liminf_{x\to\infty} P(x) < \limsup_{x\to\infty} P(x)$ *implies oscillation.*

THEOREM (iii). $\lim_{x\to\infty} P(x) = +\infty$ *implies oscillation.*

Since (2) implies the hypothesis of (iii), Theorem (iii) implies Theorem (i).

The above theorems do not apply if $P(x)$ tends to a finite limit or if $\liminf_{x\to\infty} P(x) = -\infty$; e.g., they give no information about such coefficients as $p(x) = (x\cos x - \sin x)/x^2$ or $p(x) = x^2 \sin x$. The purpose of this note is to derive oscillation criteria for certain classes of such coefficients.

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2. **Weighted averages.** The idea is that additional information about oscillation of (1) may be gained by considering *weighted* averages of $\int^x p$. Let f be a nonnegative, locally integrable function such that $\int_0^x f \not\equiv 0$; then there is an $a > 0$ such that

$$A(x) = A(f;p)(x) = \int_0^x f(t) \int_0^t p(s) ds dt \Big/ \int_0^x f(t) dt \tag{5}$$

exists on $[a, \infty)$.

THEOREM 1. *If there exists a nonnegative, locally integrable function f satisfying*

$$\int_a^\infty \left\{ f(t) \left(\int_0^t f(s) ds \right)^k \Big/ \int_0^t f^2(s) ds \right\} dt = +\infty \tag{6}$$

for some k, $0 \leqq k < 1$, and for $a > 0$

and

$$\lim_{x \to \infty} A(x) = +\infty, \tag{7}$$

then (1) *is oscillatory.*

PROOF. The proof uses ideas of Hartman. We give a proof for f continuous; the proof is easily modified for f locally integrable. Also, if convenient, we will change the lower limits of the integrals in (5) and (6), since the asymptotic behavior as $x \to \infty$ is not changed thereby.

Suppose that (1) is nonoscillatory; then, for large enough a, a solution of the Riccati equation

$$z' + z^2 + p(x) = 0 \tag{8}$$

exists on $[a, \infty)$. Integration, multiplication by f, and integration give

$$\int_a^x f(t) z(t) dt + \int_a^x f(t) \int_a^t z^2(s) ds dt = (z(a) - A(x)) \int_a^x f(t) dt. \tag{9}$$

By hypothesis, the right-hand side tends to $-\infty$; hence, for large enough x,

$$\int_a^x f(t) z(t) dt + \int_a^x f(t) \int_a^t z^2(s) ds dt < 0$$

so that

$$(10)\qquad \left(\int_a^x f(t)\int_a^t z^2(s)dsdt\right)^2 \leqq \left(\int_a^x f(t)z(t)dt\right)^2 \leqq \int_a^x f^2(t)dt\cdot\int_a^x z^2(t)dt.$$

Let $R(x)=\int_a^x f(t)\int_a^t z^2(s)dsdt$. Since, for $x\geqq b>a$, $R(x)\geqq\int_b^x f(t)dt\cdot\int_a^b z^2(t)dt$, we have from (10) that

$$f(x)\left(\int_b^x f(t)dt\right)^k\left(\int_a^b z^2(t)dt\right)^k \Big/ \int_a^x f^2(t)dt \leqq R^{k-2}(x)R'(x).$$

For $b>a$, integration now gives

$$\int_b^x \left\{f(t)\left(\int_b^t f(s)ds\right)^k \Big/ \int_a^t f^2(s)ds\right\} \leqq \frac{1}{h}\left(\frac{1}{R^h(b)}-\frac{1}{R^h(x)}\right)<\frac{1}{hR^h(b)}\qquad (h=1-k)$$

contradicting (6).

Note that (6) implies that

$$(11)\qquad \int^\infty f(t)dt=+\infty,$$

a reasonable condition for a weight function. Conversely, if f is bounded then (11) implies (6) for any k such that $0\leqq k<1$.

If (6) holds for some k on $[0, 1)$, it holds for $k=0$; but one advantage in stating the theorem for $k>0$ is that weight functions x^α $(\alpha>0)$ are permitted.

The following corollary says roughly that if $\int_0^x p$ is large enough on a large enough set, then (1) is oscillatory regardless of the behavior of $\int_0^x p$ on the rest of the half line.

COROLLARY 1. *Let* $S(x)=\{t\mid 0\leqq t\leqq x$ *and* $\int_0^t p>0\}$, *and let* $m(S(x))$ *be the measure of* $S(x)$. *If* $m(S(x))\to\infty$ *as* $x\to\infty$ *and if*

$$\frac{1}{m(S(x))}\int_{S(x)}\int^t p(s)dsdt\to\infty$$

as $x\to\infty$, *then* (1) *is oscillatory.*

PROOF. Take $f(x)$ to be 1 if $\int_0^x p>0$ and 0 otherwise, let $k=0$, and apply Theorem 1.

EXAMPLES. If $p(x) = x \sin x$, Theorems (i) and (iii) fail, but Theorem (ii) and Corollary 1 apply.

If $p(x) = x^2 \sin x$, Theorems (i), (ii), and (iii) fail, but Corollary 1 applies.

Even if no suitable weight function exists for Theorem 1, (1) may oscillate, as the following theorem implies.

THEOREM 2. *If $P(x)$ does not approach a finite limit as $x \to \infty$ and if there is a nonnegative, locally integrable function f satisfying* (6) *and*

$$\liminf_{x\to\infty} A(x) > -\infty, \tag{12}$$

then (1) *is oscillatory.*

The interesting way for $P(x)$ to fail to have a limit is

$$\liminf_{x\to\infty} P(x) = -\infty; \tag{13}$$

this is the only case not covered by Theorems (ii) and (iii).

PROOF OF THEOREM 2. First we remark that since $\int^\infty f = +\infty$ if $g(x)$ is nondecreasing in x, we have:

(a) $\int^x f(t)g(t)dt / \int^x f(t)dt$ is nondecreasing in x;

(b) if $\int^x f(t)g(t)dt / \int^x f(t)dt$ is bounded on $[a, \infty)$, so is $g(x)$.

Suppose that (1) is nonoscillatory. Via the Riccati equation we have (by (12))

$$\left[\int_a^x f(t)z(t)dt + \int_a^x f(t)\int_a^t z^2(s)dsdt\right] \Big/ \int_a^x f(t)dt$$
$$= z(a) - A(x) \leqq K \qquad (K \text{ constant}) \text{ on } \quad [b, \infty), b > a.$$

We claim that $\int_a^x f(t)\int_a^t z^2(s)dsdt / \int_a^x f(t)dt$ is bounded on $[b, \infty)$. If not, by (a), it tends to $+\infty$; and so, for large x,

$$\int_a^x f(t)z(t)dt + \frac{1}{2}\int_a^x f(t)\int_a^t z^2(s)dsdt$$
$$\leqq \left(K - \left(\int_a^x f(t)\int_a^t z^2(s)dsdt/2\int_a^x f(t)dt\right)\right)\int_a^x f(t)dt < 0.$$

Now one proceeds as in the proof of Theorem 1 to contradict (6).

So, by (b), $\int^\infty z^2 < \infty$. But Hartman [1] has shown that if (1) is nonoscillatory, then $\int^\infty z^2 < \infty$ if and only if $P(x)$ has a finite limit as $x \to \infty$. This contradiction completes the proof.

EXAMPLE. Let $p(x)$ be such that $\int_0^x p$ is 0 on $[2n, 2n+1]$ $(n = 0, 1, \cdots)$ but is sufficiently negative on $(2n+1, 2n+2)$

($n=0, 1, \cdots$) to produce $\liminf_{x\to\infty} P(x)=-\infty$. Let $f(x)$ be 1 on $[2n, 2n+1]$ ($n=0, 1, \cdots$) and 0 elsewhere so that $A(x)\equiv 0$ on $(0, \infty)$. Theorems (i), (ii), (iii), and 1 do not apply, but Theorem 2 does.

3. **The general selfadjoint case.** Corresponding to Theorems 1 and 2 are Theorems 1° and 2° for the equation

$$(1^\circ)\quad (r(x)y')'+p(x)y=0 \ (r(x)>0;\ r(x) \text{ and } p(x) \text{ continuous on } [0,\infty)).$$

Equation (6) becomes

$$(6^\circ)\qquad \int_0^\infty \left\{f(t)\left(\int_0^t f(s)ds\right)^k \Big/ \int_0^t r(s)f^2(s)ds\right\} dt = +\infty.$$

The proofs parallel those of Theorems 1 and 2 so are omitted here. The result of Hartman, needed in the proof of Theorem 2, holds for (1°) with $\int^\infty z^2$ replaced by $\int^\infty z^2/r$.

The Leighton-Wintner theorem for (1°), namely that (1°) is oscillatory if $\int^\infty p=\int^\infty 1/r=+\infty$, follows from Theorem 1° on taking $f=1/r$ and $k=0$. On the other hand, if (6°) holds for some f, then $\int^\infty 1/r=+\infty$; and (7) implies that $\limsup_{x\to\infty} \int^x p=+\infty$. Thus Theorem 1° (and Theorem 1) are interesting only in case $\int^\infty(1/r)=+\infty$, $\liminf_{x\to\infty} \int^x p<\limsup_{x\to\infty} \int^x p=+\infty$.

As with Theorem 2, the interesting case of Theorem 2° is $\liminf_{x\to\infty} P(x)=-\infty$.

References

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