COVERING THEOREMS AND THE AXIOM OF CHOICE

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Consider nonempty sets A, B and a binary relation R on A to B. For $b \in B$, $R^{-1}(b)$ is that subset of A each of whose elements is R-related to b. For $B' \subseteq B$, $R^{-1}[B']$ is the union of all $R^{-1}(b)$ where $b \in B'$. Let α be any family of subsets of A that contains the empty set and that contains $A' \cup \{a\}$ whenever $A' \in \alpha$ and $a \in A$. Let β contain all subsets of B that are related to α as follows: if $A' \subseteq A$, $B' \subseteq B$, and R' is a refinement of R that is one-one from A' onto B'; then $A' \in \alpha$ only if $B' \in \beta$. Theorem 1 of [1] can be stated as follows.

(P) *If $A' \subseteq A$ is such that whenever $A'' \subseteq A'$ and $A'' \notin \alpha$, then there exists $p \in A''$ for which $A'' \cap R^{-1}(b) \notin \alpha$ in case pRb; then whenever $B' \subseteq B$ is such that $A' \subseteq R^{-1}[B']$, there is some $B'' \subseteq B'$ for which both $B'' \in \beta$ and $A' \subseteq R^{-1}[B'']$.*

The proof of (P) is due to Menger (Theorem II_2 of [2], but misstated there) who uses the well-ordering principle and transfinite induction. Thus we have

(1) The axiom of choice implies (P).

The conditions on α are satisfied if α is the power set of A, in which case there is no $A'' \notin \alpha$, and (P) yields

(Q) *If $A' \subseteq A$, $B' \subseteq B$, and $A' \subseteq R^{-1}[B']$, then there is some $B'' \subseteq B'$ for which $A' \subseteq R^{-1}[B'']$ and $B'' \in \beta$ (where β contains a subset of B provided there is a refinement of R which establishes a one-to-one correspondence between that subset and some subset of A).*

We have, therefore, that

(2) (P) implies (Q).

THEOREM. (Q) *implies the axiom of choice.*

PROOF. Let $A = A'$ be a family of pairwise disjoint, nonempty sets. Set $B = B' =$ the union of A. Define aRb iff $b \in a$ and call β the family of all those subsets $X \subseteq B$ such that for each $a \in A$, card $(a \cap X) \leq 1$. Clearly β satisfies the condition of (Q), $A' \subseteq R^{-1}[B']$, and so (Q) yields a set B'' such that $A' \subseteq R^{-1}[B'']$ —i.e., B'' contains at least one element from each set in the disjoint family A' —and $B'' \in \beta$ —i.e., B'' contains no more than one element from each set in A'.

From (1) and (2) one obtains

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COROLLARY.
(a) *The axiom of choice is equivalent to* (P).
(b) *The axiom of choice is equivalent to* (Q).
(c) (P) *is equivalent to* (Q).

Originally, (P) was formulated as an abstract version of the topological covering theorem for separable spaces (if there is a condensation point in each uncountable subset, then every open covering of any subspace contains a countable subcovering). Similarly, in [**3**] Mickle and Rado have formulated an abstract version of the measure-theoretic covering theorems of Vitali type. They show that their theorem is equivalent to the axiom of choice. By the above results, therefore, the axiom of choice provides an abstract version for both sorts of covering theorems.

REFERENCES

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3. E. J. Mickle and T. Rado, *On covering theorems*, Fund. Math. **45** (1958), 325–331.

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