ON THE SUBRING GENERATED BY THE SYMMETRIC ELEMENTS

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J. M. Osborn [3] has characterized those rings with involution J, with 1, and such that every symmetric element has an inverse under the additional hypothesis that S^-, the subring generated by S (the set of symmetric elements), is A, the ring itself. In this paper we raise the question as to when $S^- = A$ under a weaker hypothesis and prove the following theorem. (A henceforth is such that $2A = A$ and $2x = \theta$ implies $x = \theta$ for any $x \in A$.)

THEOREM 1. *Let A be a ring with 1 and suppose that S is a simple Jordan ring under the Jordan multiplication, $s \circ t = st + ts$ for all s and t in S. Then either $S^- = A$ or $S \subset Z$, the center of A, or K, the set of skew elements, is an ideal of A with $K^2 \subset Z$, $K^3 = (\theta)$, and S is an associative ring under Jordan multiplication.*

We note that Osborn proves that under his hypothesis, either A is a division ring; the direct sum of two division rings which are anti-isomorphic, and J interchanges the summands; or S is a field under Jordan multiplication, K is an ideal of A where $K^2 = (\theta)$. A corollary to our result is

COROLLARY 1. *Let A be as in Theorem 1. Suppose further that A contains no nilpotent ideals. Then either A is simple or A is a direct sum of two simple rings which are anti-isomorphic and such that J interchanges the direct summands.*

In order to prove these results we note quickly

LEMMA 1. *If U is a proper ideal of A then $U \cap S = (\theta)$.*

This follows since $S \cap U$ is a Jordan ideal of S and hence if not zero is S itself. But $1 \in S$ implies that $U = A$.

We are ready to prove Theorem 1. Herstein [1] has shown that S^- is a Lie ideal of A and Zuev [4] has shown that either S^- is commutative (in particular $[S, S] = (\theta)$) or $I = \{u \mid u \in S^-, ua \in S^- \text{ for all } a \in A\}$ is a nonzero two-sided ideal of S^-. We next note that $[S, S] \subseteq I$. To

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observe this fact, let $s, t, u \in S$, $k \in K$; then most certainly $(st-ts)u \in S^-$, while $[s, t]k = s[t, k]+[s, k]t+\{kst-tsk\}$. As the last quantity is in S we have our desired conclusion. Moreover, $[S, S] \subseteq V = I \cap I^J$. Thus, by Lemma 1 we have either $V = A$ and hence $A = I \subset S^-$ (part of our desired conclusion) or $V \cap S = (\theta)$. So far we have shown that either $S^- = A$, $[S, S] = (\theta)$, or $[S, S] \subseteq V$, where V is an ideal such that $V \cap S = (\theta)$.

We will have almost proved Theorem 1 if we can show that the latter situation ($[S, S] \subseteq V$, V an ideal with $V = V^J$ and $V \cap S = (\theta)$) implies either $[S, S] = (\theta)$ or the latter possibility in the consequence of the theorem. We now restrict ourselves to this case.

If $u \in V$ then there exists $v \in V$ so that $u = v^J$. Thus, as a consequence of $V \cap S = (\theta)$ we have $u+v = u+u^J = \theta$, or $V \subseteq K$. V is an ideal and so for all $a \in A$, $u \in V$, $ua - a^J u = \theta$ (in particular, $[S, S] \circ K = (\theta)$). This guarantees that $[S, K]$ is a Jordan ideal of S since $[S, K] \circ S \subseteq K \circ [S, S] + [S, K]$. Thus, the simplicity of S implies that either $[S, K] = S$ or $[S, K] = (\theta)$.

We now see that the subcase $[S, K] = S$ leads to $[S, S] = (\theta)$. To this end, consider $a, b \in A$, $v \in V$. Then,

$$\theta = vab - (ab)^J v = (va - a^J v)b + a^J(vb - b^J v) + (a^J b^J - b^J a^J)v$$

or

$$[A, A]V = (\theta).$$

In particular, $[S, K][S, S] = (\theta)$ and so $[S, K] = S$ and $1 \in S$ yields $[S, S] = (\theta)$.

Thus, the subcase $[S, K] = (\theta)$ remains. Here we show that either $S \subset Z$ or the latter alternative of the theorem holds. In this case we have $[k^2, s+l] = k \circ [k, s] + [k, k \circ l] = \theta$ for all $k, l \in K$, $s \in S$. Thus, $k^2 \in Z$ for all $k \in K$. Therefore, $kl+lk = (k+l)^2 - k^2 - l^2 \in Z$ for all $k, l \in K$ or $K \circ K \subseteq Z$. Now, $[K, S] + K \circ K$ is a Jordan ideal of S. However, this is just $K \circ K$ under our hypothesis. Thus, either $K \circ K = S \subseteq Z$ (a desired conclusion) or $K \circ K = (\theta)$. The latter situation now concerns us. Now for all $a \in A$, $k \in K$, $ka - a^J k \in [S, K] + K \circ K = (\theta)$. Thus, $k, l \in K$, $a \in A$ implies $[kl, a] = k(la - a^J l) + (ka^J - ak)l = \theta$, or $K^2 \subset Z$. Under our assumptions we also note that $KA \subseteq K \circ S + [K, K] \subseteq K$; that is, K is an ideal of A. $K \circ K = (\theta)$ implies that $k^2 = \theta$ and $k \circ l = \theta$ for all $k, l \in K$. Thus, $klk = \theta$ and replacing k by $k+m$, m also in K, we have $klm + mlk = \theta$. But K is an ideal, so $(ml)k + k(ml) = mlk - klm = \theta$. Hence, $K^3 = (\theta)$. Now for all $s, t, u \in S$, $(s \circ t) \circ u = s \circ (t \circ u) + [t, [s, u]]$. The latter term being zero implies that S is an associative ring under the Jordan multiplication.

Therefore, we have shown that either $S^- = A$, $[S, S] = (\theta)$, or the third consequence in the statement of the theorem holds. What remains is to show that $[S, S] = (\theta)$ implies $S \subseteq Z$. Fix $s \in S$ and consider $[s, K]$. For all $k \in K$, $t \in S$

$$[s, k] \circ t = [s, k \circ t] - k \circ [s, t].$$

By hypothesis the latter term is zero and so $[s, K]$ is a Jordan ideal of S. Now, if $[s, K] = (\theta)$ for each $s \in S$ then $S \subseteq Z$ (as $[S, S] = (\theta)$). We show that the other alternative, $[u, K] = S$ for some $u \in S$ leads to a contradiction. As $[S, S] = (\theta)$ we conclude that $[u, [u, a]] = \theta$ for all $a \in A$. Therefore, replacing a by ab and expanding out (using the fact that $2x = \theta$ implies $x = \theta$) we obtain for all $a, b \in A$

$$(ua - au)(ub - bu) = \theta.$$

Since $1 \in S$ and $[S, S] = (\theta)$, $S^2 = S$ and so $[u, K]^2 = S^2 = S = (\theta)$, a contradiction. Therefore, $S \subseteq Z$ as desired.

The proof of Corollary 1 follows. We first note that the additional hypothesis on A implies that either $S^- = A$ or $S \subseteq Z$. Now, if A is not simple then we wish to show that the alternative of the conclusion holds. Let U be a proper nonzero ideal of A. It readily follows that $T = \{u + u^J \mid u \in U\}$ is a Jordan ideal of S. Now, if $T \neq (\theta)$ then $S = T$ or $A = U + U^J$. We show that indeed the summation is direct and U is a simple ring. By Lemma 1, $U \cap S = (\theta)$. Hence, $V = U \cap U^J$ is an ideal with the property $V \cap S = (\theta)$ as well. As before, we conclude that $V \subseteq K$, and $V^3 = (\theta)$. Hence, $V = (\theta)$ or the summation is direct. Now, let $W \neq (\theta)$ be an ideal of U. Then, $R = UWU$ is an ideal of A, contained in U, and nonzero. Else, $(AR)^3 \subseteq UWU = (\theta)$, and this is a contradiction to no nilpotent ideals by Herstein [1]. As before, $A = R + R^J$. Therefore, $U \subseteq R \subseteq W$ as $W^J \subseteq U^J$. Thus, U is simple as desired. Hence, if we show $T = (\theta)$ is impossible we are done. Now, the latter implies that $U \subseteq K$, and analogous to a previous argument $U^3 = (\theta)$. But the additional hypothesis forcing $U = (\theta)$ makes this impossible.

Corollary 2. *Let A be without nilpotent ideals. Then either $S^- = A$ or A is simple and $[A:Z] \leqq 4$ or $A = U + U^J$, where U is simple and $[U:Z] \leqq 4$.*

We have shown in Theorem 1 that either $S^- = A$ or $S \subseteq Z$. Now suppose the latter. Then given any $r \in A$ there exists $s \in S$, $k \in K$ so that $r=s+k$. Thus $r^2-2sr=k^2-s^2\in Z$, or each element in A, satisfies a quadratic equation over Z. The latter, together with Corollary 1, by the work of Kaplansky [2], yields the desired conclusion.

In addition if A is prime then A cannot be written as $U+U^J$ and so A is simple.

We assume $1\notin A$. Then certain of the results carry over and others do not. The absence of 1 does not allow us to conclude that if U, an ideal, is such that $U\supseteq S$ then $U=A$. However, we do have in this situation that both $a+a^J$ and aa^J are in U for all $u\in A$. Thus $a^2\in U$ for all $a\in A$, or $A^3\subseteq U$ (for example, $A^2=A$ implies that $A=U$). On the other hand, if $U\cap S=(\theta)$, then $V=U\cap U^J\subseteq K$ and $V^3=(\theta)$. $T=\{u+u^J \mid u\in U\}$ is either (θ) or S. If $T=(\theta)$ then $U\subseteq K$ and $U^3=(\theta)$; while $T=S$ implies that $A^3\subseteq U+U^J$. Now, if we restrict our attention to the ideal I as defined previously in relation to T we have either $S\subseteq I+I^J\subseteq S^-$, which says that S^- is an ideal or $T=(\theta)$; that is, $I\subseteq K$, $I\cap S=(\theta)$, and $I^3=(\theta)$. As in the proof of Theorem 1 this implies that either $[S, K]=S$ or $[S, K]=(\theta)$. As before, $[S, K]=S$ yields $[S, S]S=S[S, S]=(\theta)$ and $[S, S]K\subseteq[S, S]\circ K+[[S, S], K]\subseteq[S, S]$ (as $[S, S]\circ K=(\theta)$). Thus, in this case, $[S, S]$ is an ideal with $[S, S]^3=(\theta)$. On the other hand, $[S, K]=(\theta)$ yields the same argument as before. We summarize these remarks as

THEOREM 2. *Let A be a ring with involution and suppose that S is simple Jordan. Then either S^- is an ideal (of A) containing A^3, or $[S, S]$ is an ideal with $[S, S]^3=(\theta)$, or K is an ideal, $K^2\subseteq Z$, $K^3=(\theta)$ and S is an associative ring under Jordan multiplication.*

Now assume that there are no nilpotent ideals in A; then we have either S^- is an ideal or $[S, S]=(\theta)$. The latter yields, as in a previous argument, that for all $a, b\in A$, $u\in S$,

$$(ua-au)(ub-bu)=(\theta),$$

and replacing b by ba and expanding we have $\{(ua-au)A\}^2=(\theta)$. But by the hypothesis and Herstein [1] we conclude that $S\subseteq Z$. Thus, either S^- is an ideal or $S\subseteq Z$ and every $a\in A$ satisfies a quadratic equation over Z. Now if U is any proper nonzero ideal, then $S=\{u+u^J \mid u\in U\}$ and so $a+a^J=u+u^J$ for each $a\in A$ and suitable $u\in U$. Therefore $a-u\in K$ and so $A=U+K$. If $U\cap S=(\theta)$ then, under these hypotheses, $U\cap K=(\theta)$ and so the group sum is direct.

Finally, if A is prime and U is a nonzero ideal then U contains S (and hence the ideal S^-, unless $[S, S]=(\theta)$), as the other alternative $U\cap S=(\theta)$ implies $U\cap U^J=(\theta)$ or $UU^J=(\theta)$ which, by the hypothesis, is impossible.

References

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