BLASCHKE QUOTIENTS AND NORMALITY

JOSEPH A. CIMA AND PETER COLWELL

Let f be meromorphic in the unit disc D and call f normal if the family $\{f \circ S_\alpha\}$ is normal, where $\{S_\alpha\}$ represents the family of one to one conformal mappings of D onto D. Further, if $\{z_n\}$ is a sequence in D and $w=\{w_1, w_2, w_3, \cdots\}$ is a vector in l^∞ we say that $\{z_n\}$ interpolates if there is a function in H^∞ with $f(z_n)=w_n$, $n=1, 2, 3, \cdots$. A criterion that $\{z_n\}$ interpolate was given by L. Carleson [**2**] as

$$A(n) = \prod_{k=1;\,k\neq n}^{\infty} \left| (z_n - z_k)/(1 - \bar{z}_k z_n) \right| \geqq \delta > 0$$

all $n=1, 2, 3, \cdots$. Our first result states that if $\{z_n\}$ interpolates then the Blaschke product for $\{z_n\}$, written as $B(z; z_n)$, has its modulus greater than some positive number η for points not close to $\{z_n\}$.

More precisely we let $\psi(z, z_n) = |(z-z_n)/(1-\bar{z}_n z)|$ be the pseudo-hyperbolic distance in the disc of z to z_n. Then we have

THEOREM 1. *If $\{z_n\}$ interpolates and $B(z)=B(z; z_n)$ is the Blaschke product for $\{z_n\}$ then for each $\epsilon>0$ there is an $\eta>0$ such that $|B(z)| \geqq \eta$ whenever $\psi(z, z_n) \geqq \epsilon$, $n=1, 2, 3, \cdots$.*

PROOF. Assume there is a sequence $\{t_k\}$ in D and $\epsilon_0>0$ so that $\psi(t_k, z_n) \geqq \epsilon_0$ all k and n and that $B(t_k)$ tends to zero as k tends to infinity. By a result of A. T. Cargo [**1**, p. 142] we may select a subsequence of $\{t_k\}$, which we again write as $\{t_k\}$, so that the Blaschke product $A(z)=A(z; t_k)$ formed from the sequence $\{t_k\}$ has the property that

$$|A(z_n)| \geqq a > 0, \qquad n = 1, 2, 3, \cdots.$$

We have then

$$|A(z)| + |B(z)| > 0, \qquad z \in D.$$

Then as in Theorem 1 of [**3**] we can find functions $g_1(z)$ and $g_2(z)$ holomorphic in D and satisfying

$$A(z)g_1(z) + B(z)g_2(z) = 1, \qquad z \in D.$$

The sequence $\{(A(z_n))^{-1}\}$ is in l^∞ so there is an f_1 in H^∞ satisfying $f_1(z_n) = (A(z_n))^{-1}$, $n = 1, 2, 3, \cdots$. Also we have that

Received by the editors March 27, 1967.

$$(f_1(z) - g_1(z)) \cdot (B(z))^{-1} = h(z)$$

with h holomorphic in D.

Define a holomorphic function f_2 on D as follows:

$$f_2(z) = g_2(z) - h(z)A(z).$$

Then

$$f_1(z)A(z) + f_2(z)B(z) = 1, \qquad z \in D,$$

and

$$f_2(z) = (1 - f_1(z)A(z)) \cdot (B(z))^{-1}$$

is a bounded holomorphic function. Evaluating f_2 on the sequence $\{t_n\}$ shows it is unbounded, which is a contradiction, and so we have the result.

If we are given sequences of distinct points $A_1 = \{\alpha_n\}$ and $A_2 = \{\beta_n\}$ in D with corresponding Blaschke products $B_1(z) = B(z;\, \alpha_n)$ and $B_2(z) = B(z;\, \beta_n)$, we have the following theorem concerning the normality of the quotient $B_1(z) \cdot (B_2(z))^{-1}$.

THEOREM 2. *If A_1 and A_2 are disjoint interpolating sequences in D, then the meromorphic function $f(z) = B_1(z) \cdot (B_2(z))^{-1}$ is normal in D if and only if $A_1 \cup A_2$ interpolates.*

PROOF. Assume first $f(z) = B_1(z)(B_2(z))^{-1}$ is normal in D. A criterion of O. Lehto and K. I. Virtanen [**4**, pp. 55–56] states that

$$\rho(f) \leqq C\,|dz|/(1 - |z|^2), \qquad |z| < 1,$$

where ρ is the spherical derivative of f and C is a positive constant. Evaluating this inequality on the points $\{\alpha_j\}$ of A_1 yields

$$|B_2(\alpha_j)| \geqq (1/C)\,|B_1'(\alpha_j)|\,(1 - |\alpha_j|^2).$$

We denote the partial product

$$\frac{|\alpha_j|}{-\alpha_j}\left(\frac{1 - \bar{\alpha}_j z}{z - \alpha_j}\right) B_1(z) \equiv \prod_{n=1;\, n \neq j}^{\infty} \frac{-\alpha_n}{|\alpha_n|}\left(\frac{z - \alpha_n}{1 - \bar{\alpha}_n z}\right)$$

as $\prod_j(z)$ and write $B_1(z)$ as

$$B_1(z) = \frac{-\alpha_j}{|\alpha_j|}\left(\frac{z - \alpha_j}{1 - \bar{\alpha}_j z}\right) \cdot \prod_j (z).$$

The condition of L. Carleson quoted previously shows that $|\prod_j(\alpha_j)| \geqq \delta > 0$. This shows that

$$\liminf |B_2(\alpha_j)| \geqq (1/C)\delta > 0.$$

Thus $A_1 \cup A_2$ interpolates.

Now assume $A_1 \cup A_2$ interpolates. If the set of numbers

$$|f'(z)|(1-|z|^2)/(1+|f(z)|^2)$$

were unbounded there must exist a sequence $\{z_k\}$ in D such that

$$|B_1(z_k)|^2 + |B_2(z_k)|^2 \to 0.$$

as k tends to infinity. By Theorem 1 it must be the case that for some subsequence of A_1, say $\{\alpha_{k_n}\}$, we have $\psi(z_{k_n}, \alpha_{k_n})$ tends to zero. Also there must be a subsequence $\{\beta_{j_{k_n}}\}$ with $\psi(z_{j_{k_n}}, \beta_{j_{k_n}})$ tending to zero. This shows that $\psi(\alpha_{j_{k_n}}, \beta_{j_{k_n}})$ tends to zero which is a contradiction to $A_1 \cup A_2$ interpolating. The spherical derivative is bounded by a constant times $(1-|z|^2)^{-1}|dz|$ and so f is normal.

We point out that $\rho(f)=\rho(1/f)$ so that our Theorem 2 can be stated for $B_2(z)\cdot(B_1(z))^{-1}$. The referee has pointed out the following equivalence.

THEOREM 3. *If A_1 and A_2 are disjoint interpolating sequences in D, then $B_1(z)\cdot(B_2(z))^{-1}$ is normal if and only if B_1 and B_2 are in no proper ideal of H^∞.*

PROOF. If the ideal generated by B_1 and B_2 is not proper there are functions f and g in H^∞ satisfying

$$f(z)B_1(z) + g(z)B_2(z) = 1, \qquad z \in D.$$

Setting $h_1(z)=f(z)B_1(z)$ and $h_2(z)=g(z)B_2(z)$ we observe that h_1 is zero on A_1 and one on A_2. Similarly h_2 is one on A_1 and zero on A_2. Using these functions and the hypothesis that A_1 and A_2 interpolate, we see that $A_1 \cup A_2$ interpolates.

Conversely, if $A_1 \cup A_2$ is interpolating then there is an $f \in H^\infty$ with $fB_1=1$ on A_2. Hence $1-fB_1$ is divisible by B_2 and thus one is in the ideal generated by B_1 and B_2.

REFERENCES

1. G. T. Cargo, *Normal functions, the Montel property, and interpolation in H^∞*, Michigan Math. J. **10** (1963), 141–146.

2. L. Carleson, *An interpolation problem for bounded analytic functions*, Amer. J. Math. **80** (1958), 921–930.

3. J. A. Cima and G. D. Taylor, *On the equation $f_1g_1+f_2g_2=1$ in H^p*, Illinois J. Math. (to appear).

4. O. Lehto and K. I. Virtanen, *Boundary behavior and normal meromorphic functions*, Acta Math. **97** (1957), 47–65.

UNIVERSITY OF NORTH CAROLINA AND
IOWA STATE UNIVERSITY