ON THE EXISTENCE THEOREM FOR DIFFERENTIAL EQUATIONS

JOEL W. ROBBIN

In this note we show how to derive the fundamental existence theorem for ordinary differential equations as a corollary of the implicit function theorem in Banach spaces.[1] The proof of smoothness with respect to initial conditions is considerably shorter than existing proofs (see, for example [3], [4], or [5]). Throughout, a dot (i.e., $\dot{\phi}$) denotes differentiation with respect to t.

THEOREM. *Let U be an open set in a Banach space $\boldsymbol{E}$ and let $f\colon \boldsymbol{R}\times U\to\boldsymbol{E}$ be a C^r map ($r\geqq 1$). Then for each $x_0\in U$ there exists an open neighborhood V of x_0 in U, an open interval $(-\epsilon, \epsilon)$ about 0 in $\boldsymbol{R}$ and a map $\phi\colon(-\epsilon,\epsilon)\times V\to U$ such that*

(1) *ϕ is C^r;*
(2) *$\phi(0, x)=x$ for $x\in V$;*
(3) *$\dot{\phi}(t, x)=f(t,\phi(t, x))$ for $(t, x)\in(-\epsilon,\epsilon)\times V$.*

PROOF. We suppose without loss of generality that x_0 is the origin of $\boldsymbol{E}$ and that U is an open ball with center x_0. Take U_0 to be the open ball whose center is x_0 and whose radius is half the radius of U. Let I denote the closed interval $[-1, 1]\subseteq\boldsymbol{R}$. For p an integer $\geqq 0$ let $C^p(I, \boldsymbol{E})$ denote the Banach space of C^p maps from I to $\boldsymbol{E}$ (with the C^p topology), $C_0^p(I, \boldsymbol{E})$ be the (closed) subspace of $C^p(I, \boldsymbol{E})$ consisting of all $\gamma\in C^p(I, \boldsymbol{E})$ with $\gamma(0)=0$, and $C_0^p(I, U_0)$ the set of all $\gamma\in C_0^p(I, \boldsymbol{E})$ such that $\gamma(I)\subseteq U_0$. Note that $C_0^p(I, U_0)$ is open in the Banach space $C_0^p(I, \boldsymbol{E})$. D denotes the differentiation operator (see [4] or [5]) and D_j denotes partial differentiation with respect to the jth variable.

Let $F\colon \boldsymbol{R}\times U_0\times C_0^1(I, U_0)\to C^0(I, \boldsymbol{E})$ be the map defined by

$$F(a, x, \gamma)(t) = \dot{\gamma}(t) - af(at, x+\gamma(t))$$

for $a\in\boldsymbol{R}$, $x\in U_0$, $\gamma\in C_0^1(I, U_0)$ and $t\in I$. One easily verifies that F is a C^1 map between Banach spaces. (This is an especially easy instance of the so-called omega theorem of [1]. Note that the map $\gamma\to\dot{\gamma}$ is continuous linear.) The partial derivative with respect to γ at the point $a=0$, $x=x_0$, $\gamma=0$ evaluated at the "tangent vector" $\delta\in C_0^1(I, \boldsymbol{E})$ is given by

Received by the editors October 5, 1967.

[1] I am indebted to R. Abraham for suggesting this to me.

$$D_3F(0, x_0, 0)\delta(t) = \dot{\delta}(t);$$

it is clearly a toplinear isomorphism. Since $F(0, x_0, 0)=0$ we may apply the implicit function theorem [**4**, p. 265]. This yields an open neighborhood $(-2\epsilon, 2\epsilon)\times V$ of $(0, x_0)$ in $\boldsymbol{R}\times U_0$ and a C^1 map $H: (-2\epsilon, 2\epsilon)\times V \to C_0^1(I, U_0)$ such that

$$F(a, x, H(a, x)) = 0$$

for $(a, x)\in(-2\epsilon, 2\epsilon)\times V$. We define $\phi: (-\epsilon, \epsilon)\times V\to U$ by

$$\phi(t, x) = H(\epsilon, x)(t/\epsilon) + x.$$

ϕ is C^1: this follows immediately from the fact that the evaluation map $C_0^1(I, U_0)\times I\to U_0$ is C^1 (see [**1**] or [**2**, p. 25]). $\phi(0, x)=x$ since $H(\epsilon, x)\in C_0^1(I, U_0)$. Finally, since

$$\dot{\phi}(t, x) - f(t, \phi(t, x)) = (1/\epsilon)F(\epsilon, x, H(\epsilon, x))(t/\epsilon) = 0$$

it follows that ϕ is the solution curve. We have proved the theorem in the case $r=1$. The general case follows from the case $r=1$ by an easy (and standard) induction argument.

Bibliography

1. R. Abraham, *Lectures of Smale on differential topology*, Mimeographed notes, Columbia Univ., New York, 1962.

2. R. Abraham and J. Robbin, *Transversality of flows and mappings*, Benjamin, New York, 1967.

3. E. A. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, New York, 1955.

4. J. Dieudonné, *Foundations of modern analysis*, Academic Press, New York, 1960.

5. S. Lang, *Introduction to differentiable manifolds*, Interscience, New York, 1962.

University of Wisconsin