ON THE IDENTITY IN A MEASURE ALGEBRA

L. J. LARDY

1. **Introduction.** Let G be a commutative locally compact topological semigroup, and let $M(G)$ denote the complex Banach algebra of all bounded regular Borel measures on G, where the product is defined by convolution. That is, if $\mu, \nu \in M(G)$ and A is a Borel subset of G, then

$$(\mu * \nu)(A) = \int_G \int_G \varphi_A(xy) d\mu(x) d\nu(y),$$

where φ_A denotes the characteristic function of A. A general treatment of the algebras $M(G)$ appears in [**4**] and also in the survey article [**6**]. Special algebras of this type are discussed in [**3**] and [**2**].

The set of discrete measures in $M(G)$ forms a subalgebra of $M(G)$, denoted by $l_1(G)$. Of course, if the topology on G is discrete, then $l_1(G) = M(G)$. A comprehensive study of $l_1(G)$ was presented by Hewitt and Zuckerman in [**1**]. One of their results is that $l_1(G)$ contains an identity if and only if G contains a finite set of relative units. The purpose of this note is to show that these same conditions on G are necessary and sufficient for the existence of an identity in $M(G)$. It will follow that an identity for $M(G)$ must lie in $l_1(G)$.

The author is indebted to Professor A. Hudson for suggesting that the results in [**5**] could be used to simplify an earlier proof of Proposition 2.4.

2. **Relative units.** This section establishes certain properties of relative units. We begin with the definition.

DEFINITION. A subset U of G is a set of relative units for G if for every $x \in G$, there exists $u \in U$ such that $ux = x$. If no proper subset of U is a set of relative units, then we call U a minimal set of relative units for G.

Our attention shall be centered to a large extent on minimal sets of relative units for G. The semigroup consisting of the real numbers under the operation $xy = \max[x, y]$ furnishes an example of a semigroup that possesses sets of relative units, but fails to possess a minimal set of relative units. The latter fact is easily verified directly; however it is also a consequence of the following Proposition.

PROPOSITION 2.1. *Let U be a set of relative units for G. Then the following are equivalent.*

Received by the editors March 24, 1967.

(i) *U is minimal.*
(ii) *If $u, v \in U$ and $uv = v$, then $u = v$.*
(iii) *If $x \in G$, $u \in U$ and $xu = u$, then $x = u$.*

PROOF. (i) implies (ii). Let U be minimal, and suppose that $u, v \in U$ with $uv = v$. Then whenever $vx = x$ obtains, we also have $ux = u(vx) = (uv)x = vx = x$. By the minimality of U we must have $u = v$.

(ii) implies (iii). Suppose that $ux = u$ where $x \in G$ and $u \in U$. Choose $v \in U$ such that $vx = x$. Then $uv = (ux)v = u(vx) = ux = u$, and by (ii) we have $u = v$. Thus, $x = vx = ux = u$.

(iii) implies (i). If U is not minimal, then we can find an element $u \in U$ such that $U \setminus \{u\}$ is a set of relative units for G. Thus, there exists $v \in U \setminus \{u\}$ such that $uv = v$. Since u and v are distinct, (iii) is not satisfied.

COROLLARY 2.2. *If U is a minimal set of relative units for G, then each element of U is idempotent.*

COROLLARY 2.3. *If U is a minimal set of relative units, then U is contained in every set of relative units.*

PROOF. Let U and U' be sets of relative units for G and suppose that U is minimal. Given $u \in U$, there exists an element $u' \in U'$ such that $u'u = u$. For this u' there exists an element $v \in U$ such that $vu' = u'$. Hence, $u = u'u = vu'u = vu$. By Proposition 2.1, it follows that $u = v$. Thus we conclude that $U \subset U'$.

PROPOSITION 2.4. *If K is a compact set of relative units for G, then K contains a minimal set of relative units.*

PROOF. Let $K_0 = \{u \in K : u^2 = u\}$. We shall show that K_0 is a set of relative units for G. Let x be an arbitrary element in G. Choose $u_1 \in K$ such that $u_1 x = x$. Now choose $u_2 \in K$ such that $u_2 u_1 = u_1$. Then we also have $u_2 x = x$. We can inductively determine a sequence $\{u_n\}$ in K such that $u_m u_n = u_n$ for $m > n$ and $u_n x = x$. Since K is compact, the sequence $\{u_n\}$ has a limit point $u \in K$. It is not difficult to show that $u^2 = u$ and that $ux = x$. It follows that K_0 is a set of relative units for G.

We define a partial order on K_0, in the usual way, by agreeing that $u \leqq u'$ if and only if $uu' = u$. A standard argument using Zorn's lemma shows that each element of K_0 is contained in a maximal chain. By [5] every maximal chain is closed and has a maximal element.

Let U be the set that contains the maximal element u_L of every maximal chain L of K_0. We claim that U is the desired minimal set of relative units for G. Let $x \in G$; then there exists $u \in K_0$ such that

$ux=x$. Since $u\in L$ for some L, and since $u_L u=u$, we have $u_L x=x$. Thus U is a set of relative units for G. It follows from Proposition 2.1 that U is minimal. For suppose that $u_L u_{L'}=u_L$; then the maximality conditions imply that $u_L=u_{L'}$.

3. **The identity in** $M(G)$. We now apply the previous results to give conditions on G that are necessary and sufficient for the existence of an identity in $M(G)$. Before we proceed, we mention the following notational conventions. The total variation of $\mu\in M(G)$ is denoted by $|\mu|$. For $x\in G$, φ_x denotes the characteristic function of singleton $\{x\}$, δ_x denotes the unit point mass at x, and $V_x=\{y\in G: xy=x\}$.

LEMMA 3.1. *If $M(G)$ contains an identity E, then $E(V_x)=1$ for all $x\in G$.*

PROOF.

$$1 = \delta_x(\{x\}) = (E*\delta_x)(\{x\}) = \int_G\int_G \varphi_x(yz)d\delta_x(y)dE(z)$$

$$= \int_G \varphi_x(xz)dE(z) = E(V_x).$$

LEMMA 3.2. *If $M(G)$ contains an identity E, then G contains a compact set K of relative units.*

PROOF. Choose, by means of the regularity of E, a compact subset K of G such that $|E|(G)-|E|(K)<1$. Then from Lemma 3.1 we have $1\leqq|E|(V_x)=|E|(V_x\cap K)+|E|(V_x\backslash K)$. Since $|E|(V_x\backslash K)\leqq|E|(G\backslash K)<1$, we have $|E|(V_x\cap K)>0$. Thus, for each $x\in G$, $V_x\cap K$ is not empty and we conclude that K is a set of relative units.

THEOREM 3.3. *$M(G)$ contains an identity if and only if G contains a finite set of relative units.*

PROOF. If G contains a finite set of relative units, then a routine verification shows that the identity for $l_1(G)$ presented in [1] is also an identity for $M(G)$. Thus, if $U=\{u_1,\cdots,u_n\}$ then we have

$$E = \sum\{\delta_{u_i}: 1\leqq i\leqq n\} - \sum\{\delta_{u_iu_j}: 1\leqq i<j\leqq n\} + \cdots$$
$$+ (-1)^{n+1}\delta_{u_1u_2\cdots u_n}.$$

Conversely, suppose that $M(G)$ contains an identity E. By Lemma 3.2, G contains a compact set of relative units. Thus, from Proposition 2.4, it follows that G contains a minimal set U of relative units. If $u\in U$, then condition (iii) of Proposition 2.1 implies that $V_u=\{u\}$.

Thus by Lemma 3.1, $E(\{u\})=1$. Finally, the boundedness of E implies that U is finite.

REFERENCES

1. E. Hewitt and H. S. Zuckerman, *The l_1-algebra of a commutative semigroup*, Trans. Amer. Math. Soc. **83** (1956), 70–97.

2. ———, *Structure theory for a class of convolution algebras*, Pacific J. Math. **7** (1957), 913–941.

3. K. A. Ross, *The structure of certain measure algebras*, Pacific J. Math. **11** (1961), 723–737.

4. J. L. Taylor, *The structure of convolution measure algebras*, Trans Amer. Math. Soc. **119** (1965), 150–166.

5. L. E. Ward, *Partially ordered topological spaces*, Proc. Amer. Math. Soc. **5** (1954), 144–161.

6. J. H. Williamson, *Harmonic analysis on semigroups*, J. London Math. Soc. **42** (1967), 1–41.

SYRACUSE UNIVERSITY