QUOTIENTS OF COMPLETELY REGULAR SPACES

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In a previous paper [1] we gave necessary and sufficient conditions for a quotient space of a pseudo-metrizable space to be pseudo-metrizable. In this note we give a short proof of the corresponding theorem for preservation of complete regularity by quotient maps. The proof specializes in an obvious way to the pseudo-metric case and has the advantage that, unlike the proof in [1], it requires neither the use of uniformities nor the complicated construction of that paper. Moreover, we obtain an interesting explicit definition of a pseudo-metric (or, in the complete regularity case, a defining family of pseudo-metrics) for the quotient space.

For the most part the terminology here is standard. But we wish to make some things explicit. If p is a pseudo-metric for X, and if $\epsilon>0$, $x\in X$, and A, $B\subset X$, then

$$N_\epsilon[x] = N_{p,\epsilon}[x] = \{z \in X \mid p(z, x) < \epsilon\},$$
$$p(A, B) = \inf\{p(a, b) \mid a \in A, b \in B\},$$
$$N_\epsilon[A] = N_{p,\epsilon}[A] = \{z \in X \mid p(z, A) < \epsilon\}.$$

The topology on a space X defined by a family P of pseudo-metrics for X is the topology with $\{N_{p,\epsilon}[x] \mid p\in P,\ \epsilon>0,\ x\in X\}$ as subbase. (We do not require in the above definition that P separate points; so the topology generated by P need not be Hausdorff.) Recall that a topology on X is completely regular if and only if it can be defined by a family of pseudo-metrics.

THEOREM 1. *Let f be a function from a completely regular space X onto a topological space Y, and suppose that Y has the quotient topology relative to f. Then the following assertions are equivalent*:

(1) *Y is completely regular.*

(2) *There exists a family P_0 of pseudo-metrics defining the topology of X and a subbase $\mathcal{S}$ of the topology of Y such that for each $G\in\mathcal{S}$ there exists $p\in P_0$ and a set $\{\epsilon(y, p) \mid y\in G\}$ of positive real numbers satisfying*

(i) $N_{p,\epsilon(y,p)}[f^{-1}[y]]\subset f^{-1}[G]$, *if* $y\in G$,

(ii) $p(f^{-1}[y], f^{-1}[z]) \geqq \epsilon(y, p) - \epsilon(z, p)$, *if* $y, z\in G$.

(3) *There exists a family P_0 of pseudo-metrics defining the topology of X such that the topology of Y is defined by the family $Q = \{q_p \mid p\in P_0\}$*

Received by the editors March 20, 1967.

[1] This research was supported in part by the National Science Foundation (GP-6490).

of pseudo-metrics defined by

$$q_p(y, z) = \inf \sum_{i=1}^{n} p(f^{-1}[y_{i-1}], f^{-1}[y_i]),$$

where $y, z \in Y$, $y_i \in Y$ *for all* $1 \leqq i \leqq n$, *and the infimum is taken over all finite chains* $y = y_0, y_1, \cdots, y_n = z$.

REMARK. Assertion (1) implies the existence of a single family P_0 which satisfies the requirements of both (2) and (3). Also the proof of (1)$\Rightarrow$(2) requires only that f be continuous, and not necessarily that f also be a quotient map.

PROOF OF (1)$\Rightarrow$(2). Let Y be completely regular, let P, Q be families of pseudo-metrics which define the topologies of X, Y, respectively. For each $q \in Q$, let $\mathcal{S}_q$ be the topology on Y defined by q. Then let $\mathcal{S} = \bigcup \{\mathcal{S}_q \mid q \in Q\}$. For each $(p, q) \in P \times Q$, define p_q: $X \times X \to R$ by

$$p_q(x, y) = p(x, y) + q(f(x), f(y)), \quad \text{if } x, y \in X.$$

Let $P_0 = \{p_q \mid (p, q) \in P \times Q\}$. Trivially, each member of P_0 is a continuous pseudo-metric for X; so the topology on X defined by P_0 is smaller than the topology defined by P. Thus, since $p_q(x, y) \geqq p(x, y)$, for $(p, q) \in P \times Q$, and $x, y \in X$, it follows that P_0 and P define the same topology, i.e., P_0 defines the given topology on X. Now let $G \in \mathcal{S}$, say $G \in \mathcal{S}_q$, with $q \in Q$, and let $p \in P$ be arbitrary. Define

$$\epsilon(y, p_q) = q(y, Y - G), \quad \text{if } y \in G.$$

By the way q was chosen, it is trivial that each such $\epsilon(y, p_q)$ is positive. It is also easy to check (i) and (ii) of (2).

PROOF OF (2)$\Rightarrow$(3). Let P_0 and $\mathcal{S}$ be given as in (2) and let Q be defined as in (3). It is easily shown that each $q_p \in Q$ is a pseudo-metric for Y. Moreover for each $p \in P_0$, f is continuous (in fact decreases distances) if X, Y are given the topologies defined by p, q_p, respectively. Thus f is continuous relative to the topologies defined by P_0 and Q. All that remains to be shown is that the topology defined by Q is larger than the quotient topology on Y. To do this it is sufficient to show that each member of $\mathcal{S}$ is open in the topology defined by Q. So let $G \in \mathcal{S}$, and let $p \in P_0$ and $\{\epsilon(y, p) \mid y \in G\}$ be as given by (2). Then we claim that, for all $y \in G$, $q_p(z, y) < \epsilon(y, p) \Rightarrow z \in G$.

For suppose $q_p(z, y) < \epsilon(y, p)$. Then there exists a chain $y = y_0, y_1, \cdots, y_n = z$ of points of Y such that

$$(*) \qquad \sum_{i=1}^{n} p(f^{-1}[y_{i-1}], f^{-1}[y_i]) < \epsilon(y, p).$$

In particular, $p(f^{-1}[y], f^{-1}[y_1]) < \epsilon(y, p)$. This means that $p(f^{-1}[y], u) < \epsilon(y, p)$ for some $u \in f^{-1}[y_1]$. Consequently, $u \in f^{-1}[G]$ and $y_1 = f(u) \in G$.

Now apply (ii) of (2) to (*) to obtain

$$\begin{aligned}\sum_{i=2}^{n} p(f^{-1}[y_{i-1}], f^{-1}[y_i]) &< \epsilon(y, p) - p(f^{-1}[y], f^{-1}[y_1]) \\ &\leqq \epsilon(y, p) - \epsilon(y, p) + \epsilon(y_1, p) \\ &= \epsilon(y_1, p).\end{aligned}$$

Thus by repeating the argument following the inequality (*), we deduce successively that $y_1, y_2, \cdots, y_n = z$ all belong to G. We have thus proved that G is open relative to q_p, and hence is open in the topology defined by Q.

PROOF OF (3)$\Rightarrow$(1). Trivial.

A simplified (P_0 and Q in the statements of (2) and (3), and P, Q in the proof of (1)$\Rightarrow$(2) will all have only one element) version of the above argument now gives the following pseudo-metric version of Theorem 1.

THEOREM 2. *Let f be a function from a pseudo-metrizable space X onto a topological space Y, and suppose that Y has the quotient topology relative to f. Then the following assertions are equivalent*:

(1) *Y is pseudo-metrizable.*

(2) *There exists a pseudo-metric p defining the topology of X and a subbase $\mathcal{S}$ for the topology of Y such that for each $G \in \mathcal{S}$ there exists a set $\{\epsilon(y) \mid y \in G\}$ of positive real numbers satisfying*

(i) *$N_{\epsilon(y)}[f^{-1}[y]] \subset f^{-1}[G]$, if $y \in G$,*

(ii) *$p(f^{-1}[y], f^{-1}[z]) \geqq \epsilon(y) - \epsilon(z)$, if $y, z \in G$.*

(3) *There exists a pseudo-metric p defining the topology of X such that the topology of Y is defined by the pseudo-metric q defined by*

$$q(y, z) = \inf \sum_{i=1}^{n} p(f^{-1}[y_{i-1}], f^{-1}[y_i]),$$

where $y, z \in Y$, $y_i \in Y$, $1 \leqq i \leqq n$, and the infimum is taken over all finite chains $y = y_0, y_1, \cdots, y_n = z$.

REMARK. (1)$\Leftrightarrow$(2) is the main theorem of [**1**].

REFERENCE

1. C. J. Himmelberg, *Preservation of pseudo-metrizability by quotient maps*, Proc. Amer. Math. Soc. **17** (1966), 1378–1384.

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