

A NOTE ON WEAK BEZOUT RINGS

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A weak Bezout ring is defined in [1] as an integral domain with unit (not necessarily commutative) such that any two principal right ideals with nonzero intersection have sum and intersection which are again principal. In this note we weaken the condition that a ring be a weak Bezout ring by means of the following theorem which enables us to drop the condition that intersections be principal.

THEOREM. *Let K be an integral domain in which each two principal right ideals with nonzero intersection have a principal sum. Then K is a weak Bezout ring.*

The proof is an immediate consequence of the following

LEMMA. *Let K be as in the above Theorem. If $a, b \in K$ and $aK + bK = K$, there is an invertible matrix in K_2 with first row a, b .*

PROOF. If $a = 0$ or $b = 0$, the proof is clear, so take $a \neq 0, b \neq 0$. Let $ax - by = 1$. We may always take $x \neq 0, y \neq 0$. Note that $a(xa - 1) = bya$ and $b(yb + 1) = axb$ are in $aK \cap bK$. Consider the elements

$$(1) \quad b_1 = xa - 1, \quad b_2 = xb, \quad a_1 = ya, \quad a_2 = yb + 1.$$

From $b_2y = b_1x$, $b_1K \cap b_2K \neq 0$, whence $b_1K + b_2K$ is principal, say $b_1K + b_2K = b_3K$. For some c_1, c_2, r, s in K we have

$$(2) \quad b_1 = b_3c_1, \quad b_2 = b_3c_2, \quad b_1r + b_2s = b_3.$$

Defining $a_3 = a_1r + a_2s$ it follows that $ab_3 = ba_3$, whence the above equations yield the relations

$$(3) \quad ba_1 = ab_1 = ab_3c_1 = ba_3c_1, \quad ba_2 = ab_2 = ab_3c_2 = ba_3c_2.$$

Cancelling b in the appropriate equations, we obtain $a_1 = a_3c_1$ and $a_2 = a_3c_2$. Now this result and (2) may be inserted into (1) to obtain:

$$(4) \quad xa - b_3c_1 = 1, \quad xb - b_3c_2 = 0, \quad -ya + a_3c_1 = 0, \quad -yb + a_3c_2 = 1.$$

In terms of matrices, (4) becomes

$$(5) \quad \begin{bmatrix} x & -b_3 \\ -y & a_3 \end{bmatrix} \begin{bmatrix} a & b \\ c_1 & c_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

As a and b have a nonzero common right multiple each matrix is invertible.

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PROOF OF THEOREM. Let a_0, b_0 be in K with $a_0K + b_0K = dK \neq 0$. Then for some a, b, x, y in K , $a_0 = da$, $b_0 = db$, $ax - by = 1$. By the Lemma, a, b is the first row of an invertible matrix in K_2 . Let the matrix and its inverse be as in (5) above. Now let $ak_1 + bk_2 = 0$. By equating the elements of

$$\begin{bmatrix} a & b \\ b_3 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} a & b \\ b_3 & 0 \end{bmatrix} \begin{bmatrix} x & -b_3 \\ -y & a_3 \end{bmatrix} \begin{bmatrix} a & b \\ c_1 & c_2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

and cancelling b_3 we obtain $k_1 = -b_3(c_1k_1 + c_2k_2)$. Thus $aK \cap bK = ab_3K$. It follows that $a_0K \cap b_0K = a_0b_3K$.

REFERENCE

1. P. M. Cohn, *Noncommutative unique factorization domains*, Trans. Amer. Math. Soc. **109** (1963), 452-464.

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