ON CRITERIA OF DEFINABILITY

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A class $\mathbf{K}$ of relational systems is AC_δ if there exists a set S of sentences (of the language determined by the type of relational systems in $\mathbf{K}$) such that $\mathbf{K}$ consists precisely of the models of S: $\mathbf{K} = \mathrm{Mod}(S)$. The class $\mathbf{K}$ is AC if, moreover, a finite set S can be chosen here. If $\mathbf{L}$ is a class of relational systems, a class $\mathbf{K} \subseteq \mathbf{L}$ is $AC_\delta(\mathbf{L})$ or $AC(\mathbf{L})$ if there exists a class $\mathbf{K}'$ such that $\mathbf{K} = \mathbf{K}' \cap \mathbf{L}$ and $\mathbf{K}'$ is AC_δ or AC respectively. Let Π, Ω, $\mathcal{E}\mathcal{S}$, $\mathcal{I}$ be the operations of taking ultraproducts, ultralimits, elementary subsystems, and isomorphic images. Kochen [**4**] has shown

THEOREM 1. *If* $\mathbf{K} \subseteq \mathbf{L}$, $\mathbf{K}$ *is closed under* Π, Ω, $\mathcal{I}$ *and* $\mathbf{L} - \mathbf{K}$ *is closed under* Ω, *then* $\mathbf{K}$ *is* $AC_\delta(\mathbf{L})$. *If, moreover,* $\mathbf{L} - \mathbf{K}$ *is closed under* Π, *then* $\mathbf{K}$ *is* $AC(\mathbf{L})$.

Kochen's proof is based on his earlier characterization of classes AC and AC_δ (Kochen [**3**, Theorem 11.6]), which follows from the present theorem by taking $\mathbf{L}$ to be the class of all relational systems of the type under consideration. Now Kochen's proof of that characterization depended on a thorough analysis of the relationship between ultralimits and prenex normal forms. It is a first purpose of this note to give a new proof of Theorem 1, based solely on the following facts:

(1) If $\langle A_i \mid i \in I \rangle$, $\langle B_i \mid i \in I \rangle$ are sequences of relational systems and $A_i \equiv B_i$ (A_i elementarily equivalent to B_i) for every $i \in I$, then $\Pi_D \langle A_i \mid i \in I \rangle \equiv \Pi_D \langle B_i \mid i \in I \rangle$ for every ultrafilter D on I (Frayne-Morel-Scott [**1**, Corollary 2.4]).

(2) If $A \equiv B$, then A and B have isomorphic ultralimits (Kochen [**3**, Theorem 9.3]).

(3) A class $\mathbf{K}$ is AC_δ if and only if $\mathbf{K}$ is closed under Π and $\mathcal{E}$ (elementary equivalence) (Frayne-Morel-Scott [**1**, Theorem 2.13]). It should be noted that both (2) and (3) follow rather directly from Frayne's lemma.

(4) If $A \equiv B$, then B is isomorphic to an elementary subsystem of a suitable ultrapower of A (Frayne-Morel-Scott [**1**, Theorem 2.12], Kochen [**3**, Lemma 9.1]).

It is well known that Kochen's characterization of classes AC_δ is an easy consequence of (2) and (3).

For a proof of Theorem 1, assume $\mathbf{K} \subseteq \mathbf{L}$ and define $\mathbf{C}(\mathbf{K})$ by $A \in \mathbf{C}(\mathbf{K})$ if there exists $A' \in \mathbf{K}$ and $A' \equiv A$; define $\mathbf{C}(\mathbf{L} - \mathbf{K})$ sim-

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ilarly. If $\mathbf{K}$ is closed under Π, it follows from (1) that $\mathbf{C}(\mathbf{K})$ is closed under Π. Hence $\mathbf{C}(\mathbf{K})$ is AC_δ by (3). If both $\mathbf{K}$ and $\mathbf{L}-\mathbf{K}$ are closed under Ω and $\mathbf{K}$ is closed under $\mathcal{G}$, it follows from (2) that $\mathbf{C}(\mathbf{K})$ and $\mathbf{C}(\mathbf{L}-\mathbf{K})$ are disjoint. Hence $\mathbf{C}(\mathbf{K})\cap\mathbf{L}=\mathbf{K}$, i.e. $\mathbf{K}$ is $AC_\delta(\mathbf{L})$. If, moreover, $\mathbf{L}-\mathbf{K}$ is closed under Π, also $\mathbf{C}(\mathbf{L}-\mathbf{K})$ is AC_δ by symmetry. It can be assumed now that neither $\mathbf{K}$ nor $\mathbf{L}-\mathbf{K}$ are empty. If $\mathrm{Th}(\mathbf{K})$ and $\mathrm{Th}(\mathbf{L}-\mathbf{K})$ are the sets of sentences true in $\mathbf{K}$ and $\mathbf{L}-\mathbf{K}$ respectively, then $\mathbf{C}(\mathbf{K})\cap\mathbf{C}(\mathbf{L}-\mathbf{K})=0$ shows that $\mathrm{Th}(\mathbf{K})\cup\mathrm{Th}(\mathbf{L}-\mathbf{K})$ is inconsistent, while $\mathrm{Th}(\mathbf{K})$ and $\mathrm{Th}(\mathbf{L}-\mathbf{K})$ alone are consistent. Hence there exist finite and nonempty subsets $S_0\subseteq\mathrm{Th}(\mathbf{K})$, $S_1\subseteq\mathrm{Th}(\mathbf{L}-\mathbf{K})$ such that $S_0\cup S_1$ is inconsistent, whence $\mathrm{Mod}(S_0)\cap\mathrm{Mod}(S_1)=0$. Therefore $\mathrm{Mod}(S_0)\cap\mathbf{L}=\mathbf{K}$, i.e. $\mathbf{K}$ is $AC(\mathbf{L})$.

An even simpler argument, making use of (4) instead of (2), yields

THEOREM 2. *If* $\mathbf{K}\subseteq\mathbf{L}$, $\mathbf{L}$ *is closed under* $\mathcal{G}$ *and* $\mathbf{K}$ *is closed under* Π, $\mathcal{G}$, $\mathcal{E}\mathcal{S}\restriction\mathbf{L}$, *then* $\mathbf{K}$ *is* $AC_\delta(\mathbf{L})$. *If, moreover,* $\mathbf{L}-\mathbf{K}$ *is closed under* Π, *then* $\mathbf{K}$ *is* $AC(\mathbf{L})$.

Kochen ([**3**, Theorem 12.1], [**4**, Theorem 3]) has used Theorem 1 in order to give a mathematical characterization of definable model functions, of which Beth's theorem on definability is an immediate consequence. Here a model function U on a class $\mathbf{L}$ of relational systems assigns to every $A\in\mathbf{L}$ a relational system $U(A)=\langle A, R_A\rangle$, having one additional new relation R_A of a fixed arity. U is definable with respect to $\mathbf{L}$ if there exists a formula of the language determined by $\mathbf{L}$ which, for every $A\in\mathbf{L}$, defines the relation R_A in terms of A. Now Theorem 2 can be applied in order to obtain

THEOREM 3. *Let U be a model function on* $\mathbf{L}$, *and let* $\mathbf{L}$ *be closed under* $\mathcal{G}$ *and* Π. *U is definable with respect to* $\mathbf{L}$ *if and only if U commutes with the operations* Π, $\mathcal{G}$ *and* $\mathcal{E}\mathcal{S}$.

The proof is essentially that of Kochen's Theorem 12.1 in [**3**]. Namely, let the relations R_A be n-ary. Let $\mathbf{L}'$ be the class of all relational systems $\langle A, \alpha\rangle$, where $A\in\mathbf{L}$ and α is a sequence of n elements of the set $s(A)$ underlying A; $\mathbf{L}'$ then is described by a type that extends the type of $\mathbf{L}$ by n new constants. Let $\mathbf{K}$ be the class of all $\langle A, \alpha\rangle$ in $\mathbf{L}'$ such that $\alpha\in R_A$. U will be definable with respect to $\mathbf{L}$ if and only if $\mathbf{K}$ is $AC(\mathbf{L}')$. Now with $\mathbf{L}$, also $\mathbf{L}'$ is closed under $\mathcal{G}$. It follows from Kochen's proof that U commutes with Π and $\mathcal{G}$ if and only if $\mathbf{K}$ and $\mathbf{L}'-\mathbf{K}$ are closed under Π and $\mathcal{G}$. Further, if U is definable then it commutes with $\mathcal{E}\mathcal{S}$. Assume now that U commutes with $\mathcal{E}\mathcal{S}$ and let $\langle B, \beta\rangle\in\mathbf{L}'$ be an elementary subsystem of $\langle A, \alpha\rangle\in\mathbf{K}$. Then $\beta=\alpha$, and B is an elementary subsystem of A. Therefore,

$\langle B, R_B\rangle$ is an elementary subsystem of $\langle A, R_A\rangle$, and in particular $R_B = s(B)^n \cap R_A$. But then $\beta = \alpha$ and $\alpha \in R_A$ implies $\beta \in R_B$, i.e. $\langle B, \beta\rangle \in \mathbf{K}$.

Theorem 3 has applications in Hoehnke's work [2] on the mathematical characterization of definable maps between classes of relational systems.

It follows from well-known results of Keisler's that sufficient belief in GCH would enable us to omit any assumptions concerning $\mathcal{E}\mathfrak{E}$ in Theorem 2 and Theorem 3.

References

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