INDICES OF MAXIMAL SUBGROUPS OF INFINITE SYMMETRIC GROUPS

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Let M be an infinite set with cardinal number $|M| = X$, $S(X, Y) = \{s: s$ is a permutation of M with $|\text{spt } s| < Y\}$, where spt $s = \{m \in M: s(m) \neq m\}$. If X is a cardinal number, denote its successor by X^+; the smallest infinite cardinal shall be denoted by d. Higman [3] has shown that $S(X, d)$ has only one proper subgroup of index less than X, the alternating subgroup $A(X)$, and $A(X)$ has no proper subgroups of index less than X. Gaughan [2] has extended these results showing that if $d < Y \leqq X^+$, then $S(X, Y)$ has no proper subgroups of index less than X.

One might conjecture that any proper maximal subgroup of $S(X, Y)$ would have minimal index, namely X. It is the purpose of this paper to demonstrate that such is not the case by constructing examples with index greater than X. It will also be shown that all intransitive proper maximal subgroups of $S(X, Y)$ do have index X.

Hereafter, maximal shall mean proper maximal. Let M be partitioned into P and Q with $|M| = X \geqq d$, $|P| = X$, $|Q| = Z$, $0 < Z \leqq X$, $d \leqq Y \leqq X^+$. Let $J(Z) = S(Q) \cdot S(P)$, $J(Y, Z) = J(Z) \cap S(X, Y)$. It has been shown [1] that if $Z < d$, then $J(Y, Z)$ is a maximal subgroup of $S(X, Y)$. In fact every intransitive maximal subgroup of $S(X, Y)$ is of this form. Let $s \in S(X, X^+)$, and define $P_s = \{x \in P: s(x) \in Q\}$ and $Q_s = \{y \in Q: s(y) \in P\}$. For each $s \in S(X, X^+)$ define the *transfer index of* $s = T(s) = \max\{|P_s|, |Q_s|\}$. Now define $L(Y, Z)$ as follows: If $d \leqq Z < Y \leqq X^+$, $Z < X$, then $L(Y, Z) = \{s \in S(X, Y): T(s) < Z\}$. If $d < Y \leqq Z \leqq X$ and Y has an immediate predecessor Y^-, then $L(Y, Z) = \{s \in S(X, Y): T(s) < Y^-\}$. It has been shown [1] that $L(Y, Z)$ is a (transitive) maximal subgroup of $S(X, Y)$.

THEOREM 1. *Let H be an intransitive maximal subgroup of $S(X, Y)$. Then $[S(X, Y): H] = X$.*

PROOF. By [1], $H = J(Y, Z)$ with $0 < Z < d$. By Gaughan [2], $[S(X, Y): H] \geqq X$. Since H is intransitive, $A(X) \not\subseteq H$, so by maximality, $S(X, Y) = A(X) \cdot H$, and $[S(X, Y): H] \leqq |A(X)| = X$.

Let H be any subgroup of a group G. By $\mathrm{Cl}(H)$ is meant the set of all conjugates of H in G, $\mathrm{Cl}(H) = \{gHg^{-1}: g \in G\}$. It is well known that $[G: N(H)] = |\mathrm{Cl}(H)|$, where $N(H)$ is the normalizer of H in G.

Received by the editors October 1, 1965 and, in revised form, April 16, 1967.

A suitable choice of cardinals X, Y, and Z will now be made to construct $H=L(Y, Z)$ with $[S(X, Y): H]>X$.

LEMMA 1. *Let $d \leqq Z \leqq X$. Let C be a class of subsets of M of cardinality Z such that if $Q_1, Q_2 \in C$, $Q_1 \neq Q_2$, then $|Q_1-Q_2|=Z$. Then $|C| \leqq X^Z$, and there is such a class C_0 such that $|C_0|=X^Z$.*

PROOF. The set of all subsets of M of cardinality Z has cardinality X^Z. Thus $|C| \leqq X^Z$. Let $M=\bigcup_{x \in M} Q_x$ disjointly such that for all $x \in M$, $|Q_x|=Z$. Also let $M=\bigcup_{a \in A} P_a$ disjointly such that $|A|=Z$ and for all $a \in A$, $|P_a|=X$. Let F be the family of nonvoid subsets of M such that for each $D \in F$ and for each $a \in A$, $|D \cap P_a|=1$. F has the property that $D_1 \not\subseteq D_2$ if $D_1 \neq D_2$. Furthermore $|F|=X^Z$, and $|D|=Z$ if $D \in F$. Let $E_D=\bigcup_{x \in D} Q_x$, and $C_0=\{E_D: D \in F\}$. Now C_0 has the desired property, and $E_{D_1}=E_{D_2}$ if and only if $D_1=D_2$, so $|C_0|=|F|=X^Z$.

THEOREM 2. *Let M be partitioned into P and Q as before with $d<Z<Y \leqq X^+$, $Z<X$. Then $[S(X, Y): L(Y, Z)]=X^Z$.*

PROOF. By [1], $L(Y, Z) = J(Y, Z) \cdot S(X, Z)$. Hence for any $s \in S(X, Y)$, $sL(Y, Z)s^{-1}=s[J(Y, Z) \cdot S(X, Z)]s^{-1}=[sJ(Y, Z)s^{-1}] \cdot S(X, Z)=[S(s(Q), Y) \cdot S(s(P), Y)] \cdot S(X, Z)$. Thus $sL(Y, Z)s^{-1} \neq rL(Y, Z)r^{-1}$ if and only if $|s(Q)-r(Q)|=Z$ and $|r(Q)-s(Q)|=Z$. By Lemma 1, there are X^Z subsets of M of cardinality Z having this property and at most X^Z such subsets. Since $Z<Y$, there are X^Z permutations in $S(X, Y)$ mapping Q onto distinct members of this class of subsets of M, hence $|\mathrm{Cl}(L(Y, Z))|=X^Z$. By the maximality and nonnormality of $L(Y, Z)$, $N(L(Y, Z)) = L(Y, Z)$. Thus $[S(X, Y): L(Y, Z)]=|\mathrm{Cl}(L(Y, Z))|=X^Z$.

LEMMA 2. *There are infinite cardinals Z and X such that $Z<X$ and $X<X^Z$.*

PROOF. Let Z be an infinite cardinal, A a well-ordered set of cardinality Z, $\{Z_a\}_{a \in A}$ a family of cardinal numbers such that $Z_a>Z$ for each $a \in A$, and if $a, b \in A$ with $a<b$, then $Z_a<Z_b$. Let f be a one-to-one function from A into A such that $a<f(a)$ for each $a \in A$. Then by the theorem of Koenig and Zermelo, $X=\sum_{a \in A} Z_a<\prod_{a \in A} Z_{f(A)} \leqq X^Z$.

THEOREM 3. *There are cardinal numbers X and Y, $Y \leqq X^+$, such that $S(X, Y)$ contains maximal subgroups with index larger than X.*

PROOF. By Lemma 2, choose X, Y, and Z such that $Z<Y \leqq X^+$, $Z<X$, and $X<X^Z$. By Theorem 2, $[S(X, Y): L(Y, Z)]=X^Z>X$.

Theorem 3 is somewhat limited by the choice of X and Z. However, for $S(X, X^+)$, maximal subgroups of index greater than X can be demonstrated for any choice of $X \geqq d$. Let $s \in S(X, X^+)$ and define $P_s' = P - P_s$ and $Q_s' = Q - Q_s$. For each $s \in S(X, X^+)$ define the *remainder index of* $s = R(s) = \max \{ |P_s'|, |Q_s'| \}$. Now define $L(X^+, X) = \{ s \in S(X, X^+) : T(s) < X \text{ or } R(s) < X \}$. It has been shown [1] that $L(X^+, X)$ is a (transitive) maximal subgroup of $S(X, X^+)$.

THEOREM 4. *For* $X \geqq d$, $[S(X, X^+) : L(X^+, X)] > X$.

PROOF. Partition P into P_1 and P_2 such that $|P_1| = |P_2| = X$. By Lemma 1 choose a collection C of subsets of P_1 such that $|C| = X^X = 2^X$, and if $A, B \in C$ with $A \neq B$, then $|A - B| = X$. Similarly choose a collection D of subsets of Q with the same properties. Let f be a one-to-one function from C onto D. For each $A \in C$ choose a corresponding $s \in S(X, X^+)$ such that $s(A) = f(A)$, $s(P - A) = P - A$, $s(f(A)) = A$, $s(Q - f(A)) = Q - f(A)$. Note that $T(s) = R(s) = X$, so $s \notin L(X^+, X)$. Let F denote the family of functions so defined. Let s and $r \in F$, $s \neq r$, s and r correspond to A and $B \in C$ respectively. Now $P_{r^{-1}s} \supseteq A - B$, so $X \geqq |P_{r^{-1}s}| \geqq |A - B| = X$. Thus $T(r^{-1}s) = X$. Also $P'_{r^{-1}s} \supseteq (P - A) - B = P - (A \cup B) \supseteq P_2$, so $X \geqq |P'_{r^{-1}s}| \geqq |P_2| = X$. Thus $R(r^{-1}s) = X$. Hence $r^{-1}s \notin L(X^+, X)$, so r and s determine different cosets of $L(X^+, X)$ in $S(X, X^+)$. Thus $2^X = |S(X, X^+)| \geqq [S(X, X^+) : L(X^+, X)] \geqq |F| = |C| = 2^X$, so $[S(X, X^+) : L(X^+, X)] = 2^X > X$.

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