A NOTE ON A THEOREM OF GANEA, HILTON AND PETERSON

C. S. HOO

Introduction. Let X be a space. We are interested in the question whether or not the loop space ΩX and the suspension ΣX are homotopy commutative, that is whether or not nil $X \leqq 1$, conil $X \leqq 1$ respectively. Let $i: X \flat X \to X \vee X$ be the fibre of the inclusion $j: X \vee X \to X \times X$. Let $\nabla: X \vee X \to X$ be the folding map. Then in [3], Ganea, Hilton and Peterson proved the following

THEOREM 1. *Let X be 1-connected. Then* nil $X \leqq 1$ *if and only if* $\nabla i = 0$.

Dually, let $q: X \times X \to X \wedge X$ be the cofibre of the inclusion j, and let $\triangle: X \to X \times X$ be the diagonal map. Let $e': X \wedge X \to \Omega\Sigma(X \wedge X)$ be the canonical imbedding. Then in [3], the authors also proved

THEOREM 2. *Let X be o-connected. Then* conil $X \leqq 1$ *if and only if* $e' q \triangle = 0$.

This paper represents an attempt to understand these theorems. Let $c: \Omega(X \vee X) \times \Omega(X \vee X) \to \Omega(X \vee X)$ be the commutator map. We shall define below a map $\bar{c}: \Sigma(\Omega X \times \Omega X) \to X \vee X$ obtained from c. Applying the co-Hopf construction, we have a map $H(\bar{c}): \Omega\Sigma(\Omega X \times \Omega X) \to \Omega(X \flat X)$. Then we prove

THEOREM 3. $c = \Omega(\nabla i) H(\bar{c}) e': \Omega X \times \Omega X \to \Omega X$, *the commutator map.*

We observe, of course, that the condition for nil $X \leqq 1$ is precisely $c = 0$. Dually, let $c': \Sigma(X \times X) \to \Sigma(X \times X) \vee \Sigma(X \times X)$ be the cocommutator map. This gives a map $\bar{c}': X \times X \to \Omega(\Sigma X \vee \Sigma X)$. The Hopf construction then gives a map $J(\bar{c}'): \Sigma(X \wedge X) \to \Sigma\Omega(\Sigma X \vee \Sigma X)$. Let $e: \Sigma\Omega(\Sigma X \vee \Sigma X) \to \Sigma X \vee \Sigma X$ be the map having $1_{\Omega(\Sigma X \vee \Sigma X)}$ as its adjoint. Let us denote the cocommutator product $\Sigma X \to \Sigma X \vee \Sigma X$ by c' also. The condition for conil $X \leqq 1$ is precisely $c' = 0$. We prove

THEOREM 4. $c' = eJ(\bar{c}')\Sigma(q\triangle): \Sigma X \to \Sigma X \vee \Sigma X$.

We work in the category of spaces with base point and having the homotopy type of countable CW-complexes. For simplicity, we shall frequently use the same symbol for a map and its homotopy class.

Received by the editors April 18, 1967.

1. Let A, B be spaces. We have the fibration $A \flat B \xrightarrow{i} A \vee B \xrightarrow{j} A \times B$. We can find a map $\chi: \Omega(A \times B) \to \Omega(A \vee B)$ such that $(\Omega j)\chi \simeq 1_{\Omega(A \times B)}$. In fact we can take $\chi = \Omega(i_A p_A) + \Omega(i_B p_B)$ where p_A, p_B are the projections of $A \times B$ onto the factors and $i_A: A \to A \vee B$, $i_B: B \to A \vee B$ are the inclusions. The exact sequence of the fibration now shows that there exists a unique element $[g] \in [\Omega(A \vee B), \Omega(A \flat B)]$ such that $1_{\Omega(A \vee B)} = (\Omega i)g + \chi(\Omega j)$.

Now for any space X and a map $f: X \to A \vee B$ we can form the map $H(f) = g(\Omega f): \Omega X \to \Omega(A \flat B)$. We shall call this the co-Hopf construction. The element $[H(f)]$ is the unique element of $[\Omega X, \Omega(A \flat B)]$ satisfying $[\Omega f] = (\Omega i)_\#[H(f)] + [\chi \Omega(jf)] = (\Omega i)_\#[H(f)] + [\Omega(i_A \pi_A f)] + [\Omega(i_B \pi_B f)]$ where $\pi_A: A \vee B \to A$, $\pi_B: A \vee B \to B$ are induced by the projections onto the factors.

For spaces X, Y we have a bijection $\tau: [\Sigma X, Y] \to [X, \Omega Y]$ which takes each map to its adjoint. Suppose X is a given space. We have a projection $p: \Sigma\Omega X \to X$ such that $\tau(p) = 1_{\Omega X}$. Let $p_1 = i_1 p$, $p_2 = i_2 p$ where $i_1, i_2: X \to X \vee X$ are the injections in the first and second copies of X respectively. Let $c: \Omega(X \vee X) \times \Omega(X \vee X) \to \Omega(X \vee X)$ be the commutator map. Then we can form the map $\bar{c} = \tau^{-1}\{c(\tau(p_1) \times \tau(p_2))\}: \Sigma(\Omega X \times \Omega X) \to X \vee X$. It is now easily verified that $\nabla \bar{c} = \tau^{-1}(c)$. The co-Hopf construction, applied to $\bar{c}$, gives an element $H(\bar{c}): \Omega\Sigma(\Omega X \times \Omega X) \to \Omega(X \flat X)$. Let $e': \Omega X \times \Omega X \to \Omega\Sigma(\Omega X \times \Omega X)$ be such that $e' = \tau(1_{\Sigma(\Omega X \times \Omega X)})$. It is easily seen that $\Omega(\tau^{-1}(c))e' = c: \Omega X \times \Omega X \to \Omega X$, the commutator map. Since $\nabla \bar{c} = \tau^{-1}(c)$, Theorem 3 follows immediately from

THEOREM 5. $\Omega(\nabla \bar{c}) = \Omega(\nabla i)H(\bar{c}): \Omega\Sigma(\Omega X \times \Omega X) \to \Omega X$.

PROOF. $H(\bar{c})$ satisfies $\Omega\bar{c} = (\Omega i)H(\bar{c}) + \Omega(i_1\pi_1\bar{c}) + \Omega(i_2\pi_2\bar{c})$ where $\pi_1, \pi_2: X \vee X \to X$ are induced by the projections onto the factors, and $i_1, i_2: X \to X \vee X$ are the imbeddings in the first and second copies of X respectively. We have $\Omega(\nabla\bar{c}) = \Omega(\nabla i)H(\bar{c}) + \Omega(\nabla i_1\pi_1\bar{c}) + \Omega(\nabla i_2\pi_2\bar{c})$. Let ϕ be the loop multiplication on ΩX and μ the loop inverse. Then a simple check shows that $\tau(\nabla i_1\pi_1\bar{c}) = \phi\{\phi(1 \times *)\Delta \times \phi(1 \times *)\Delta\mu\}\Delta r_1$ where Δ is the diagonal map and $r_1: \Omega X \times \Omega X \to \Omega X$ is the projection onto the first factor. Since $\phi(1 \times *)\Delta \simeq 1$ and $\phi(1 \times \mu)\Delta \simeq *$, we have $\tau(\nabla i_1\pi_1\bar{c}) = 0$. Hence $\nabla i_1\pi_1\bar{c} = 0$. Similarly $\nabla i_2\pi_2\bar{c} = 0$. It follows then that $\Omega(\nabla\bar{c}) = \Omega(\nabla i)H(\bar{c})$.

Theorems 1 and 3 are the immediate

COROLLARY. *Let X be 1-connected. If $\Omega(\nabla i) = 0$, then $\nabla i = 0$.*

REMARK. In [**3**], it is shown that there exist maps a, b such that $ba = 1$, $ib = \bar{c}$. It is clear from the above that $H(\bar{c}) = \Omega b$.

2. We now dualise. Let $p_1, p_2: X\times X\to X$ be the projections, and let $e_i = e'p_i$ where $e': X\to\Omega\Sigma X$ is the canonical imbedding. Let c' be the cocommutator map $\Sigma(X\times X)\to\Sigma(X\times X)\vee\Sigma(X\times X)$. Let $\bar{c}' = \tau\{(\tau^{-1}(e_1)\vee\tau^{-1}(e_2))c'\}: X\times X\to\Omega(\Sigma X\vee\Sigma X)$. Then $\bar{c}'\triangle = \tau(c')$ where $\triangle$ is the diagonal map.

Let A, B be spaces. We consider the cofibration $A\vee B\xrightarrow{j} A\times B\xrightarrow{q} A\wedge B$. There exists a map $p: \Sigma(A\times B)\to\Sigma(A\vee B)$ such that $p(\Sigma j)\simeq 1_{\Sigma(A\vee B)}$. The exact sequence of the cofibration now shows that $(\Sigma q)^\#$ is a monomorphism. Dual to the above, we now see that there exists a unique element $[d]\in[\Sigma(A\wedge B), \Sigma(A\times B)]$ satisfying $1_{\Sigma(A\times B)} = d(\Sigma q)+(\Sigma j)p$.

Given a map $f: A\times B\to X$ we can now define $J(f) = (\Sigma f)d: \Sigma(A\wedge B)\to\Sigma X$. We shall call $J(f)$ the map obtained from f by the Hopf construction. The element $[J(f)]$ is the unique element satisfying $[\Sigma f] = (\Sigma q)^\#[J(f)]+[\Sigma(fj)p] = (\Sigma q)^\#[J(f)]+[\Sigma(fjp_A)]+[\Sigma(fjp_B)]$ where $p_A, p_B: A\times B\to A\vee B$ are induced by the projections onto the first and second coordinates respectively. We can now consider the element $J(\bar{c}'): \Sigma(X\wedge X)\to\Sigma\Omega(\Sigma X\vee\Sigma X)$. We have $J(\bar{c}')\Sigma(q\triangle)$, $\Sigma(\bar{c}'\triangle): \Sigma X\to\Sigma\Omega(\Sigma X\vee\Sigma X)$. Let $e: \Sigma\Omega(\Sigma X\vee\Sigma X)\to\Sigma X\vee\Sigma X$ be such that $\tau(e) = 1_{\Omega(\Sigma X\vee\Sigma X)}$. Let $c': \Sigma X\to\Sigma X\vee\Sigma X$ be the cocommutator map. It is now easily checked that $e\Sigma(\tau(c')) = c'$. Since $\bar{c}'\triangle = \tau(c')$. Theorem 4 follows immediately from

THEOREM 6. $\Sigma(\bar{c}'\triangle) = J(\bar{c}')\Sigma(q\triangle): \Sigma X\to\Sigma\Omega(\Sigma X\vee\Sigma X)$.

PROOF. The proof is completely dual to that of Theorem 5, and we shall omit it.

REMARK 1. In [3], it is shown that we can find maps a', b' such that $b'a' = 1$, $a'e'q = \bar{c}'$. It is easily seen that $J(\bar{c}') = \Sigma(a'e')$.

REMARK 2. Theorems 3 and 4 give other conditions for nil $X\leqq 1$, conil $X\leqq 1$ respectively, namely whenever some combination of factors in the factorizations of c, c' is null-homotopic.

REFERENCES

1. I. Berstein and T. Ganea, *Homotopical nilpotency*, Illinois J. Math. **5** (1961), 99–129.

2. I. Berstein and P. J. Hilton, *On suspensions and comultiplications*, Topology **2** (1963), 73–82.

3. T. Ganea, P. J. Hilton and F. P. Peterson, *On the homotopy-commutativity of loop-spaces and suspension*, Topology **1** (1962), 133–141.

4. P. J. Hilton, *Homotopy theory and duality*, Gordon and Breach, New York, 1965.

UNIVERSITY OF ALBERTA