ON THE ROOTS OF SPECTRAL OPERATORS

CONSTANTIN APOSTOL

Introduction. Let X be a Banach space and T a linear bounded operator acting in X. It is known that if T is invertible and T^n is a spectral operator for some natural number n, then T is spectral (see [9]).

Our principal result (Theorem 2.1) is a sufficient (and evidently necessary) condition for T to be a spectral operator when T^n is spectral.

1. Throughout we shall denote by X a Banach space and by $\mathcal{L}(X)$ the algebra of all linear bounded operators acting in X. N resp. Λ will be the set of all natural resp. complex numbers. The field of all Borel subsets of Λ will be denoted by $\mathcal{B}$. For $n \in N$, $\sigma \in \mathcal{B}$ we shall write $\sigma^{1/n}$ for the set $\{\lambda: \lambda \in \Lambda, \lambda^n \in \sigma\}$. For any $0 \leqq r$ we shall denote $\Lambda_r = \{\lambda: \lambda \in \Lambda, r < |\lambda|\}$.

LEMMA 1.1. *Let $\sigma, \tau \in \mathcal{B}$ and $n, m \in N$ be two relatively prime numbers. If we have $|\arg \lambda - \arg \mu| < 2\pi/mn$ for any $\lambda, \mu \in \sigma \cup \tau$, $\lambda\mu \neq 0$, then the equality $(\sigma^m)^{1/m} \cap (\tau^n)^{1/n} = \sigma \cap \tau$ holds.*

PROOF. Evidently we have $\sigma \cap \tau \subset (\sigma^m)^{1/m} \cap (\tau^n)^{1/n}$. Let now $\zeta \in (\sigma^m)^{1/m} \cap (\tau^n)^{1/n}$. It follows that there exist $\lambda \in \sigma$, $\mu \in \tau$ such that $\lambda^m = \zeta^m$, $\mu^n = \zeta^n$. If $\zeta = 0$ then $\zeta = \lambda = \mu$ thus $\lambda \in \sigma \cap \tau$. Let us suppose that $\zeta \neq 0$. Then we have $|\zeta| = |\lambda| = |\mu|$ and there exist two integers k, r such that $|k| < m$, $|r| < n$ and $\arg \zeta - \arg \lambda = 2k\pi/m$, $\arg \zeta - \arg \mu = 2r\pi/n$.

It follows that $|\arg \lambda - \arg \mu| = (2\pi/mn)|kn - rm| < 2\pi/mn$; thus $|kn - rm| = 0$. Since m and n are relatively prime the last inequality implies $k = r = 0$ thus $\zeta = \lambda = \mu \in \sigma \cap \tau$ which proves our lemma.

Let us now suppose that T^m and T^n are spectral operators, $m, n \in N$, with the spectral measures F and respectively G.

For any $0 \leqq r$ the subspace $X_r = F(\Lambda_{r^m})X$ is ultra-invariant for T (that is invariant for all operators commuting with T) as a consequence of the properties of F. For any $0 < r$ the operator $T|X_r$ is invertible and since $(T|X_r)^m = T^m|X_r$ is spectral, it follows that $T|X_r$ is spectral [9]. Let E_r be the spectral measure of $T|X_r$.

LEMMA 1.2. *Let $m, n \in N$ be two relatively prime numbers such that T^m and T^n are spectral operators and $0 \leqq \alpha < \beta \leqq 2\pi$, $\beta - \alpha \leqq 2\pi/mn$.*

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If $A=\{\lambda: \lambda\in\Lambda\setminus\{0\},\ \alpha\leqq\arg\lambda<\beta\}$, *then the map* E_A, *defined by the equation*

$$E_A(\sigma) = F((\sigma\cap A)^m)G((\sigma\cap A)^n),$$

is a homomorphic map of $\mathfrak{B}$ *into a Boolean algebra of projection operators in* X *with the identity* $E_A(A)$. $E_A(\sigma)$ *is strongly countably additive as a function of* σ.

PROOF. It is easily seen that we have $F(\Lambda_{r^m})=G(\Lambda_{r^n})$ for any $0\leqq r$. Now by Theorem 2.4.1 of [3], the operators $T^m|X_r$, $T^n|X_r$ are spectral since X_r is an ultra-invariant subspace for T and henceforth for T^m and T^n. Also by the same theorem the spectral measures of these operators are $F|X_r$ and $G|X_r$.

Now if $0<r$, the operator $T|X_r$ is spectral, and by Lemma 6 [5], the spectral measure of $(T|X_r)^m$, $(T|X_r)^n$ is $\sigma\to E_r(\sigma^{1/m})|X_r$ and respectively $\sigma\to E_r(\sigma^{1/n})|X_r$.

Using the equality $T^k|X_r=(T|X_r)^k$ for any $k\in N$ and the uniqueness of the spectral measure of a spectral operator we obtain for any $\sigma,\tau\in\mathfrak{B}$

$$E_r([(\sigma\cap A)^m]^{1/m})F(\Lambda_{r^m}) = F((\sigma\cap A)^m)F(\Lambda_{r^m}),$$
$$E_r([(\tau\cap A)^n]^{1/n})F(\Lambda_{r^m}) = G((\tau\cap A)^n)F(\Lambda_{r^m}).$$

Now by Lemma 1.1 it follows that

$$E_r(\sigma\cap\tau\cap A)F(\Lambda_{r^m}) = F((\sigma\cap A)^m)G((\tau\cap A)^n)F(\Lambda_{r^m}),$$

thus

$$E_A(\sigma) = \lim_{r\to 0} E_r(\sigma\cap A)F(\Lambda_{r^m}).$$

Evidently E_A is a homomorphism and $E_A(A)$ is the identity of the range of E_A.

Now if $\sigma=\bigcup_{k=0}^{\infty}\sigma_k$, $\sigma_k\cap\sigma_j=\emptyset$, $k\neq j$, we have for any $x\in X$

$$\begin{aligned} E_A(\sigma)x &= \sum_{k=0}^{\infty} F((\sigma_k\cap A)^m)G((\sigma\cap A)^n)x \\ &= \sum_{k=0}^{\infty} F((\sigma_k\cap A)^m)G((\sigma_k\cap A)^n)x = \sum_{k=0}^{\infty} E_A(\sigma_k)x \end{aligned}$$

which proves the countable additivity of E_A.

2. Further m and n will be two fixed relatively prime numbers, $m, n\in N$, $2\leqq m<n$.

For any $k=0, 1, \cdots, mn$, we shall denote

$$A_k = \{\lambda : \lambda \in \Lambda\backslash\{0\}, 2k\pi/(nm+1) \leqq \arg\lambda < (2k+1)\pi/(nm+1)\}.$$

THEOREM 2.1. *If T^m and T^n are spectral operators then T is a spectral operator.*

PROOF. Let $Y=F(\{0\})X$, $X_k=E_{A_k}(A_k)X$, $k=0, 1, \cdots, mn$. We have:

$$F(\{0\})E_{A_k}(A_k) = \lim_{r\to 0} F(\{0\})E_r(A_k)F(\Lambda_{r^m}) = 0,$$

$$E_{A_k}(A_k)E_{A_j}(A_j) = \lim_{r\to 0} E_r(A_k)E_r(A_j)F(\Lambda_{r^m}) = 0, \qquad k \neq j,$$

and

$$\sum_{k=0}^{mn} E_{A_k}(A_k) = \lim_{r\to 0} \sum_{k=0}^{mn} E_r(A_k)F(\Lambda_{r^m}) = \lim_{r\to 0} F(\Lambda_{r^m}) = F(\Lambda\backslash\{0\}).$$

It follows that X is the direct sum

$$X = Y \oplus \bigoplus_{k=0}^{mn} X_k, \qquad T = (T|\,Y) \oplus \bigoplus_{k=0}^{mn} (T|\,X_k).$$

We have $\sigma(T|\,Y)\subset\{0\}$; thus $T|\,Y$ is a spectral operator. Also $E_{A_k}|\,X_k$ is a resolution of the identity for $T|\,X_k$. Indeed, if we denote $E_k=E_{A_k}|\,X_k$, we have evidently $E_k(\sigma)(T|\,X_k)=(T|\,X_k)E_k(\sigma)$ and also $\sigma((T|\,X_k)^m|\,E_k(\sigma)X_k)\subset(\bar{\sigma}\cap\bar{A}_k)^m$, $\sigma((T|\,X_k)^n|\,E_k(\sigma)X_k)\subset(\bar{\sigma}\cap\bar{A}_k)^n$.

By the spectral mapping theorem [4] and Lemma 1.1 we obtain $\sigma((T|\,X_k)|\,E_k(\sigma)X_k)\subset\bar{\sigma}$.

Using Lemma 1.2 it results that $T|\,X_k$ is a spectral operator (see [4]). Thus T is a spectral operator as the direct sum of spectral operators.

THEOREM 2.2. *In the hypothesis of Theorem* 2.1, *if T^m is of scalar type then T is a spectral operator of finite type. Let S be its scalar part and Q its nilpotent part. Then we have $SQ=QS=0$, $Q^m=0$.*

PROOF. By the preceding theorem we have $T=S+Q$ where S is of scalar type and Q a generalised nilpotent. Let E be the spectral measure of T. Since T^m is scalar we have $T^m=S^m$, and $T|\,E(\Lambda_r)X$ is also scalar [9]. From the equality $TF(\Lambda_r)=SF(\Lambda_r)+QF(\Lambda_r)$ it results that $QF(\Lambda_r)=0$ is the radical part of $T|\,F(\Lambda_r)$. Consequently $Q=QF(\{0\})$.

It follows that $SQ=QS=SF(\{0\})Q=0$, $S^m=T^m=S^m+Q^m$. Thus $Q^m=0$.

COROLLARY 2.3 (OF THEOREM 2.2). *If* 0 *is not an eigenvalue of* T *then* T *is a scalar operator.*

PROOF. By Theorem 2.2, $SQ=0$, and since 0 is not an eigenvalue of T and $S^mQ=T^mQ=0$, it follows that $Q=0$, $T=S$.

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ACADÉMIE DE LA REPUBLIQUE SOCIÉTÉ DE ROUMAINE, BUCAREST