# A DECOMPOSITION RELATIVE TO CONVEX SETS

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1. The purpose of this note is to prove a decomposition theorem which asserts that, given a closed convex set $C$ in a Hilbert space $H$, each element $u \in H$ can be uniquely decomposed as the sum of an element $x \in H$ and the closest-point projection of $x$ on $C$. Moreover, $x$ depends continuously on $u$ and can be determined by an iterative procedure. We also prove a theorem on the solution by iteration of monotone operator equations.

2. In what follows, $H$ will denote a real or a complex Hilbert space with inner product $\langle u, v\rangle$. Recall that an operator $T: D \to H$ is called *monotonic* on a domain $D \subset H$ if for all $u, v \in D$, $\mathrm{Re}\langle u-v, Tu-Tv\rangle \geqq 0$; $T$ is called *strongly monotonic* if the zero in this inequality is replaced by $m\|u-v\|^2$, where $m$ is some positive number. For the original contributions to the theory of monotone operators as well as for a survey of related literature and applications to differential and integral equations, see for instance [**1**], [**2**], [**3**], [**5**], [**6**]. Let $T: H \to H$ be a continuous monotone operator. Then the equation $x+Tx=y$ has a unique solution for each $y \in H$ and the solution depends continuously on $y$ (see [**3**]). The monotonicity hypothesis can be weakened. The following theorem is due to Browder [**1**, Theorem 4].

THEOREM 1. *Let $G$: $H \to H$ be a continuous mapping such that for all $u$, $v \in H$,*

$$\mathrm{Re}\langle Gu - Gv, u - v\rangle \geqq m(\max\{\|u\|, \|v\|\})\|u - v\|^2, \tag{1}$$

*where $m(t)$ is a positive nonincreasing function of $t$ such that*

$$\int_1^\infty m(t)dt = +\infty. \tag{2}$$

*Then $G$ is one-to-one, onto, and its inverse is continuous.*

We now prove

THEOREM 2. *Let $Q$ be an operator (not necessarily linear) from $H$ into $H$ with the property that $Qu=\theta$ if and only if $u=\theta$. Let $G$: $H \to H$ satisfy the conditions*

$$\|QGu - QGv\| \leqq M(\max\{\|u\|, \|v\|\})\|u - v\| \tag{3}$$

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*and*

$$\text{(4)} \qquad \operatorname{Re}\langle QGu - QGv, u - v\rangle \geqq m(\max\{\|u\|, \|v\|\})\|u - v\|^2,$$

*where* $m(t)$ *is a positive nonincreasing function such that* (2) *and* (5) *hold and* $M(t)$ *is a positive nondecreasing function. Let*

$$\text{(5)} \qquad r = 2 \sup\{t : tm(t) \leqq \|QG\theta\|\} < \infty,$$

*let* $\beta$ *be any number in the range*[1]

$$0 < \beta < m^2(r)/M^2(r),$$

*and let a and b be the (positive) roots of the equation*

$$M^2(r)\alpha^2 - 2m(r)\alpha + \beta = 0.$$

*Then starting with the initial approximation* $x_0 = \theta$, *the iterative process*

$$\text{(6)} \qquad x_{n+1} = x_n - \alpha_n QGx_n, \qquad n = 0, 1, \cdots,$$

*where* $a \leqq \alpha_n \leqq b$, *converges to the unique solution* $x^*$ *of* $Gx = \theta$ *and*

$$\|x_n - x^*\| \leqq \tfrac{1}{2}(1 - \beta)^{n/2} r.$$

PROOF. The existence and uniqueness of the solution follow from Theorem 1.

From Schwarz's inequality we get

$$\|QGx - QG\theta\| \, \|x\| \geqq \operatorname{Re}\langle x, QGx - QG\theta\rangle \geqq m(\|x\|)\|x\|^2$$

and hence

$$\|QGx\| \geqq \|QGx - QG\theta\| - \|QG\theta\| \geqq m(\|x\|)\|x\| - \|QG\theta\|.$$

Thus

$$m(\|x^*\|)\|x^*\| \leqq \|QG\theta\|.$$

Let $r$ be as defined in (5). Then $\|x^*\| \leqq r/2$. Now consider the iteration (6). Taking $x_0 = \theta$ we get $\|x^* - x_0\| \leqq r/2$. We shall show by induction that for $\alpha_n$ in a certain range, all the iterates will lie in $\|x - x^*\| \leqq r/2$. Assume that $x_1, \cdots, x_n$ satisfy $\|x - x^*\| \leqq r/2$. Then

$$\begin{aligned} \|x_{n+1} - x^*\|^2 \leqq [1 &- 2\alpha_n m(\max\{\|x_n\|, \|x^*\|\}) \\ &+ \alpha_n^2 M^2(\max\{\|x_n\|, \|x^*\|\})]\|x_n - x^*\|^2 \\ \leqq [1 &- 2\alpha_n m(r) + \alpha_n^2 M^2(r)]\|x_n - x^*\|^2. \end{aligned}$$

Thus if there exists a $\beta$, $0 < \beta < 1$, such that

[1] Note that (3) and (4) imply $m(t) \leqq M(t)$; hence $\beta < 1$.

$$(7) \qquad 1 - 2\alpha_n m(r) + \alpha_n^2 M^2(r) \leqq 1 - \beta$$

for *all* $n$, then $x_{n+1} \in S(\theta, r)$ and

$$\|x_{n+1} - x^*\|^2 \leqq (1-\beta)^{n+1}\|x_0 - x^*\|^2 \leqq (1-\beta)^{n+1} r^2/4.$$

Hence by induction $x_n \in S(x^*, r/2)$ for all $n$. To complete the proof, note that (7) holds under the assumptions of the theorem.

REMARK 1. Results analogous to Theorems 1 and 2 may be stated for equations of the type

$$(8) \qquad x + \lambda T x = y$$

where $T$ is a mapping from $H$ into $H$, $y$ is a fixed element in $H$ and $\lambda$ is a scalar. For example, if we require that $\lambda T$ be monotonic and continuous, then the operator $G$ defined by $Gx = x + \lambda Tx - y$ is continuous and strongly monotonic since for any $u, v \in H$,

$$\operatorname{Re}\langle Gu - Gv, u - v\rangle \geqq \|u - v\|^2.$$

Under these conditions, (8) has a unique solution for each $y$.

3. **A decomposition theorem.** *Let $C$ be a closed convex set in a Hilbert space $H$ and let $P$ denote the "projection" operator on $C$. This is the operator which assigns to each $x \in H$ its closest point $Px \in C$, i.e.,*

$$\|Px - x\| = \inf\{\|y - x\| : y \in C\}.$$

It is well known that $P$ exists and is single-valued (e.g., [**4**]). $P$ is also called the closest-point map on $C$. We shall show in Theorem 3 that each $u \in H$ can be uniquely decomposed as the sum of an element $x \in H$ and its projection $Px \in C$; moreover $x$ depends continuously on $u$ and can be determined by an iterative process. We first prove the following.

LEMMA. *The closest-point map $P$ on a closed convex set $C$ in a Hilbert space $H$ is monotonic. $P$ is not strongly monotonic if $C \neq H$.*

PROOF. It is easy to show that a point $z \in C$ is the closest point to $y \notin C$ if and only if

$$(9) \qquad \operatorname{Re}\langle x - z, z - y\rangle \geqq 0 \quad \text{for all } x \in C.$$

Note that $x_t = tx + (1-t)z \in C$ for $0 \leqq t \leqq 1$ and thus $\|y - x_t\|^2 - \|y - z\|^2 \geqq 0$. Expansion of the left side of this inequality leads to (9). Thus for any $x, y \in H$ we have $\operatorname{Re}\langle x - Px, Px - Py\rangle \geqq 0$ and $\operatorname{Re}\langle Py - y, Px - Py\rangle \geqq 0$. Adding terms we get $\operatorname{Re}\langle x - y - (Px - Py), Px - Py\rangle \geqq 0$. Thus

$$(10) \qquad \operatorname{Re}\langle x - y, Px - Py\rangle \geqq \|Px - Py\|^2 \geqq 0,$$

which proves the monotonicity of $P$.

To prove the second part of the lemma, suppose $C \neq H$ and choose $x \notin C$. Let $y = 2x - Px$. Then for all $z \in C$,

$$\|x - Px\| \leqq \|x - \tfrac{1}{2}(Px + z)\|$$

or

$$\|y - Px\| \leqq \|y - z\|.$$

Hence $Py = Px$ and

$$\operatorname{Re}\langle x - y, Px - Py\rangle = 0.$$

Thus, if $P$ is strongly monotonic,

$$\operatorname{Re}\langle x - y, Px - Py\rangle \geqq m\|x - y\|^2, \qquad m > 0, \tag{11}$$

we must have $y = x$. This implies $x = Px$. But this is a contradiction since $x \notin C$.

THEOREM 3. *Let $C$ be a closed convex set in $H$. For each $u \in H$ there exists a unique $x \in H$ such that $u - x$ is the point in $C$ closest to $x$, and $x$ depends continuously on $u$. Furthermore, if $a$ and $b$ are the roots of the equation $\alpha^2 - \alpha + q = 0$, where $0 < q < 1/2$, then the sequence*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n(u - Px_n), \qquad n = 0, 1, \cdots, \tag{12}$$

*where $a \leqq \alpha_n \leqq b$, converges to $x$ starting from an arbitrary initial approximation $x_0 \in H$.*

PROOF. Consider the equation $x + Px = u$, where $P$ is the closest-point map on $C$ and $u \in H$. By the lemma, $P$ is monotonic on $H$. Furthermore, we know that $P$ is continuous and distance shrinking:

$$\|Px_1 - Px_2\| \leqq \|x_1 - x_2\| \quad \text{for any } x_1, x_2 \in H.$$

(This also follows from inequality (10).) Thus the first part of the theorem follows from Theorem 1 where $G = I + P$ and $m = 1$. This part of the theorem also follows directly from a simple argument based on the contraction principle applied to the mapping $T_\lambda = (1 - \lambda)I + \lambda(u - P)$, where $0 < \lambda < 1/2$. Indeed for $\lambda \neq 0$, every fixed point of $T_\lambda$ is a solution of the equation $x + Px = u$ and conversely. An easy computation shows that $T_\lambda$ is a strict contraction for $0 < \lambda < 1/2$. The second part follows as a special case of Theorem 2 by noting that in this case $Gx = x + Px - u$, $Q = I$, $m = 1$, and $M \leqq 2$.

REMARK 2. In the iteration (12), we have tacitly assumed that we can constructively "project" points on $C$. Since this can be usually done only for some classes of convex sets, our procedure is not completely constructive. See [7], [8] for some aspects of approximating the closest-point map.

REMARK 3. It is known (e.g., [4]) that the closest-point map exists and is single-valued for every closed convex set in a strictly convex normed linear space if and only if the space is reflexive. In view of this, it may be interesting to determine whether one can obtain a characterization of strictly convex reflexive normed linear spaces in terms of the convergence property stated in Theorem 3.

Note that it is possible to give a simple expression for the decomposition valid in any strictly convex normed space, namely, $x=u-P(u/2)$. To show that this is the desired $x$, one need only show that $P(x)=P(u/2)$, which result is implicit in the proof of the second part of the Lemma.

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