J-NOETHERIAN INTEGRAL DOMAINS WITH 1 IN THE STABLE RANGE

WILLIAM J. HEINZER

Let R be a commutative ring with identity. In [**1**, p. 29], Bass has shown that if the maximal spectrum of R is noetherian of combinatorial dimension d and if R' is a finite R-algebra, then $d+1$ defines a stable range for $\mathrm{GL}(R')$. Estes and Ohm in [**2**] have considered some topics related to Bass's theorem mainly for the case when $R'=R$. In particular, they prove, for any integer $n \geqq 0$, the existence of an integral domain D such that the maximal spectrum of D has dimension n and 1 is in the stable range of D. They raise the question of whether there exists a domain D with these properties for which the maximal spectrum of D is noetherian. Our purpose is to give an affirmative answer to this question.

If A is an ideal of R then we let $J(A)$ denote the intersection of the collection of maximal ideals of R containing A and $J=\{\text{ideals } A \text{ of } R \mid J(A)=A\}$. We say that R is J-noetherian if the ideals of J satisfy the ascending chain condition. This is equivalent to the statement that the maximal spectrum of R is noetherian. The dimension of the maximal spectrum of R is n provided there is a chain $P_0 < P_1 < \cdots < P_n$ of prime ideals of R which are in the set J but no chain of longer length. In this case, we say that R has J-dimension n. A positive integer t is said to be in the stable range of R if for $s \geqq t$ whenever $a_1, a_2, \cdots, a_{s+1}$ are elements of R such that $(a_1, a_2, \cdots, a_{s+1})=R$, then there exist $b_1, \cdots, b_s \in R$ such that $(a_1, a_2, \cdots, a_{s+1}) = (a_1+b_1a_{s+1}, \cdots, a_s+b_sa_{s+1})$. We can now state our result as follows.

THEOREM. *For each positive integer n there is a J-noetherian domain D which has* 1 *in the stable range and has J-dimension n.*

Our examples are obtained by using an existence theorem due to Jaffard [**3**, p. 78]. Jaffard has shown that if G is a lattice-ordered abelian group then there is an integral domain D which has G as its group of divisibility. His construction is carried out by using the group ring $B(G)$ of G with respect to an arbitrary field F. The elements of $B(G)$ may be regarded as formal sums $\sum_{i=1}^{n} a_iX^{g_i}$ where $a_i \in F$ and $g_i \in G$. We wish now to observe that Jaffard's construction

Received by the editors August 10, 1967.

gives, in fact, a Bezout[1] domain D which has 1 in the stable range.[2] The domain D is obtained by defining a map ϕ from $B(G)^*$, the nonzero elements of the domain $B(G)$, into G. We can write an element of $B(G)^*$ uniquely in the form $\sum_{i=1}^n a_i X^{g_i}$ where the g_i are distinct elements of G and the a_i are nonzero elements of the field F. Then $\phi(\sum_{i=1}^n a_i X^{g_i}) = \inf\{g_i\}$. We extend ϕ to E^*, the nonzero elements of the quotient field of $B(G)$, by defining $\phi(p/q) = \phi(p) - \phi(q)$ for p, $q \in B(G)^*$. If $D^* = \{y \in E^* \mid \phi(y) \geqq 0\}$ then $D = D^* \cup \{0\}$. To show that 1 is in the stable range of D and that D is Bezout we show that for $p, q \in E^*$ there is an element r of D such that $pD + qD = (p + rq)D$. Since $\phi(p) = \phi(X^{\phi(p)})$ we have $p = X^{\phi(p)} u_1$ where u_1 is a unit of D. Similarly $q = X^{\phi(q)} u_2$ where u_2 is a unit of D. If $\phi(p) \leqq \phi(q)$, then we choose $r = 0$. Since $\phi(q/p) = \phi(q) - \phi(p) \geqq 0$ we have $q \in pD$ and hence $pD + qD = pD$. If $\phi(p) \nleqq \phi(q)$ then we set $r = u_1/u_2$. Since u_1 and u_2 are units of D, $\phi(r) = \phi(u_1) - \phi(u_2) = 0$ and $r \in D$. We have $p + rq = u_1 X^{\phi(p)} + u_1 X^{\phi(q)} = u_1(X^{\phi(p)} + X^{\phi(q)})$. And because $\phi(p) \neq \phi(q)$ we have $\phi(p + rq) = \phi(u_1) + \phi(X^{\phi(p)} + X^{\phi(q)}) = 0 + \inf\{\phi(p), \phi(q)\}$. Hence $\phi(p/(p+rq))$ and $\phi(q/(p+rq))$ are positive and $pD + qD = (p + rq)D$. It follows that D is a Bezout domain. Moreover, if $(a_1, a_2, \cdots, a_{s+1}) = D$, then there exist elements b_i of D such that $(a_i, a_{s+1}) = (a_i + b_i a_{s+1})$ for $i = 1, \cdots, s$. Hence $(a_1, \cdots, a_{s+1}) = (a_1 + b_1 a_{s+1}, \cdots, a_s + b_s a_{s+1})$ and 1 is in the stable range of D.

Let G^+ denote the set of positive elements of G. By an ideal of G we will mean a nonempty subset I of G^+ with the following properties: (1) $0 \notin I$, (2) if $a \in I$ and $b > a$, then $b \in I$, (3) if a, $b \in I$ then $\inf\{a, b\} \in I$. If, in addition, the complement of I in G^+ is closed under addition then we say that I is a prime ideal. With each ideal I of G we associate the subset $\phi^{-1}(I) \cup \{0\}$ of D. It is straightforward to check that this gives a one-to-one inclusion preserving correspondence between the ideals of G and the proper integral ideals of the Bezout domain D. Moreover, I is a prime ideal of G if and only if $\phi^{-1}(I) \cup \{0\}$ is a prime ideal of D.

We proceed to construct a lattice-ordered abelian group G which has the following properties:

1. There is a chain $I_0 < I_1 < \cdots < I_n$ of prime ideals of G but no chain of longer length.

2. Every prime ideal of G is an intersection of maximal ideals.

3. For each ideal I of G there are only finitely many prime ideals of G which are minimal with respect to the property of containing I.

It follows from our previous observations that a domain D con-

---

[1] We say that D is a *Bezout domain* if every finitely generated ideal of D is principal.

[2] For the proof of this fact and for several other helpful suggestions concerning this paper, I wish to thank Jack Ohm.

structed by means of Jaffard's theorem which has G as its group of divisibility will have Krull dimension $n+1$, each proper prime ideal of D will be an intersection of maximal ideals, and each ideal of D will have only finitely many minimal prime divisors. Since D is finite dimensional this last property implies that D is J-noetherian [2]. Moreover, D will have J-dimension either n or $n+1$. If G has infinitely many minimal primes then (0) is an intersection of maximal ideals of D and $\dim_J D=n+1$; otherwise $\dim_J D=n$.

We will say that a group G which satisfies property 1 above has dimension n.

Our construction of a group G which has properties 1, 2, and 3 is based on the following two lemmas.

LEMMA 1. *Let* $\{H_\alpha\}$ *be a family of lattice-ordered abelian groups and let* H *be the weak direct sum of the* H_α*'s, where* H *is ordered by defining* $\{a_\alpha\} \geqq \{b_\alpha\}$ *if and only if* $a_\alpha \geqq b_\alpha$ *for each* α. *Let* $p_\alpha\colon H \to H_\alpha$ *be the canonical projection homomorphism. Then each prime ideal* Q *of* H *is of the form* $p_\alpha^{-1}(Q_\alpha)\cap H^+$ *where* Q_α *is a prime ideal of some* H_α *and* H^+ *is the set of positive elements of* H.

Using properties of the weak direct sum and the definition of a prime ideal in H the proof of Lemma 1 is straightforward and will be omitted.

It follows from Lemma 1 and the fact that elements in the weak direct sum have only finitely many nonzero coordinates that if $\{H_\alpha\}$ is a collection of lattice-ordered abelian groups such that each H_α satisfies properties 1, 2 and 3 above, then $H=\sum_\alpha H_\alpha$ also satisfies these properties. Moreover, if there are infinitely many α then H has infinitely many minimal primes.

LEMMA 2. *Suppose that* H *is a lattice-ordered abelian group which satisfies properties* 1, 2, *and* 3 *stated above. Assume in addition that* H *has infinitely many minimal primes. Let* T *be a totally ordered archimedean group and let* $K=T\oplus H$ *where we order* K *by defining* $(a, h) \geqq (a', h')$ *if and only if* $a>a'$ *or* $a=a'$ *and* $h\geqq h'$. *Then* K *is a lattice-ordered abelian group having dimension* $n+1$ *and* K *satisfies properties* 2 *and* 3 *stated above.*

PROOF. We note first that $Q=\{(a, h)\in K \mid a>0\}$ is the unique minimal prime ideal of K and that every ideal of K compares with Q. It is clear that Q is a prime ideal; and that Q is minimal follows from the fact that if $a, a'\in T$ with $a>0$, then for some positive integer n, $na>a'$. If $(a, h)\in K^+-Q$ then $a=0$ so $(a, h) < q$ for any $q\in Q$. Hence any ideal containing (a, h) contains Q, and it follows that every ideal of K compares with Q.

If I is an ideal of K which properly contains Q then let $I_H = \{h \in H \mid (0, h) \in I\}$. I_H is an ideal of H and I is a prime ideal of K if and only if I_H is a prime ideal of H. The mapping which associates I with I_H is a one-to-one inclusion preserving correspondence between the ideals of K which properly contain Q and the ideals of H. Since I is prime if and only if I_H is prime, we conclude that each nonminimal prime of K is an intersection of maximal ideals and that the ideals of K satisfy property 3. Finally, the fact that H has infinitely many minimal primes implies that Q is the intersection of the primes of K which properly contain Q. Hence K also satisfies property 2. This completes the proof of Lemma 2.

By starting with a totally ordered archimedean group and applying Lemmas 1 and 2 we can now obtain for any positive integer n a lattice-ordered abelian group G_n which satisfies properties 1, 2 and 3 and has a unique minimal prime. The domain D_n constructed by means of Jaffard's theorem which has G_n for its group of divisibility is J-noetherian, has 1 in the stable range, and has J-dimension n. Moreover, the fact that D_n is a Bezout domain and hence is Prüfer implies that D_n has the same prime ideal structure as any Kronecker function ring D_n^* of D_n [**4**]. In fact each ideal of D_n^* is the extension of an ideal of D_n and each ideal of D_n is the contraction of its extension. It follows that D_n^* is also J-noetherian and has J-dimension n. This gives an affirmative answer to the question raised in [**2**] concerning the existence of J-noetherian Kronecker function rings of arbitrary J-dimension.

References

1. H. Bass, *K-theory and stable algebra*, Inst. Hautes Etudes Sci. Publ. Math. No. 22 (1964), 5–60.

2. D. Estes and J. Ohm, *Stable range in commutative rings*, J. Algebra **7** (1967), 343–362.

3. P. Jaffard, *Les systèmes d'idéaux*, Dunod, Paris, 1960.

4. W. Krull, *Beiträge zur Arithmetik kommutativer Integritätsbereiche*. I, Math Z. **41** (1936), 544–577.

LOUISIANA STATE UNIVERSITY