DOMAINS OF UNIQUENESS FOR PARABOLIC EQUATIONS[1]

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1. **Introduction.** This paper concerns uniqueness questions for boundary value problems involving second order parabolic equations

$$Lu \equiv \sum a_{ij}(x, t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum a_i(x, t) \frac{\partial u}{\partial x_i} - \frac{\partial u}{\partial t} = 0, \tag{1}$$

as well as certain extensions of the well-known maximum principle. In the usual type of boundary value problem, the domain of the solution lies in the upper half plane $t > 0$ and part of its boundary lies on the hyperplane $t = 0$. The data prescribed on the latter part are referred to as initial data. The present paper, on the other hand, treats domains extending indefinitely in the $t \to -\infty$ direction. Thus instead of initial-value problems, we consider "generalized steady-state problems" (see [2]) or "problems without initial conditions." A particular case of such a domain is the cylinder $\Omega \times (-\infty, \infty)$, where Ω is a bounded domain in x-space. Some existence and uniqueness questions for boundary value problems in such a domain were treated in a previous paper [2] by the author. Another special case is the complement of such a cylindrical domain; here some obvious results can be obtained by combining the maximum principle developed in our Theorem 4 with the technique of Meyers and Serrin [4]. (See also [3] for certain results in one space dimension when the solution is periodic in time.)

In the present paper, however, the domain of the solution (which we hereafter denote by $\mathfrak{D}$) is generally allowed to be noncylindrical; the goal in fact is to obtain geometric properties of $\mathfrak{D}$ which guarantee uniqueness within the class of bounded solutions.

We shall usually require no regularity properties of the coefficients of L other than local boundedness, and for our general result (Theorem 3) we assume only that L is locally parabolic. We shall be concerned with functions satisfying $Lu \geqq 0$. The well-known standard weak maximum principle holds for such functions if the derivatives appearing in the operator are continuous; we shall designate these functions by the term *subsolutions*.

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Let us call $\mathfrak{D}$ a *domain of uniqueness* for a specific operator L if the following *maximum principle* holds:

$$\sup_{\mathfrak{D}\cap\{t<\tau\}} u(x, t) = \sup_{\partial\mathfrak{D}\cap\{t<\tau\}} u(x, t),$$

for every τ and every bounded subsolution u in $\mathfrak{D}$.

We assume throughout that each cross-section $\mathfrak{D}\cap\{t=\tau\}$ is contained in a sphere of finite radius $R(\tau)$. Therefore in proving that a certain domain is a domain of uniqueness, we need only verify that it satisfies the definition for large enough $(-\tau)$. For if the condition in the definition is satisfied for $\tau<-K$, say, then the standard maximum principle yields the verification for all other τ.

It will be shown that if $R(\tau)$ does not grow too fast as $\tau\rightarrow-\infty$, then $\mathfrak{D}$ is a domain of uniqueness. For example for the equation

$$\Delta u - u_t = 0,$$

this is true if $R^2(\tau)\leqq k|\tau|\log\log|\tau|$ for some constant $k<1$. However, an exampleshows this is not true for any k if the factor $\log\log|\tau|$ is replaced by $\log|\tau|$. Essentially the same result is true for all uniformly parabolic operators (Theorems 1 and 2). However, sufficient conditions for $\mathfrak{D}$ to be a domain of uniqueness will be given for the general case.

Regarding boundary value problems, the question here is uniqueness. The other question of existence of a bounded solution of the equation $Lu=0$ satisfying bounded boundary values on $\partial\mathfrak{D}$ is easily handled if one makes the following

Assumption A.

(i) The coefficients of L are Hölder continuous;

(ii) For every τ the initial value problem

$Lu=0$ in $\mathfrak{D}\cap\{t>\tau\}$,
smooth boundary data given on $\partial\mathfrak{D}\cap\{t>\tau\}$
and initial data on $\mathfrak{D}\cap\{t=\tau\}$ can be solved.

Namely, one solves the initial-boundary value problem with zero initial data on $t=\tau$, then lets $\tau\rightarrow-\infty$. Schauder estimates and the usual maximum principle insure that the solution approaches a limit, which will be a bounded solution on all of $\mathfrak{D}$. Existence for inhomogeneous equations could be treated in a manner analogous to that in [2].

Similar considerations yield the following proposition:

If $\mathfrak{D}$ is a domain of uniqueness, L and $\mathfrak{D}$ satisfy Assumption A, *and $\mathfrak{D}'\subset\mathfrak{D}$, then $\mathfrak{D}'$ is also a domain of uniqueness.*

PROOF. If $\mathfrak{D}'$ is not a domain of uniqueness, then there exists a subsolution u assuming some positive values, but nonpositive on $\partial\mathfrak{D}'$. Define $v(x, t) = \max[0, u(x, t)]$, and extend v to be zero in $\mathfrak{D}-\mathfrak{D}'$. For each τ, consider the solution u_τ of $Lu=0$ in $\mathfrak{D}\cap\{t>\tau\}$ vanishing on $\partial\mathfrak{D}$ and equal to v for $t=\tau$. By the Schauder estimates, as $\tau\to-\infty$ u_τ converges to a solution in $\mathfrak{D}$ vanishing on the boundary. By the standard maximum principle $u_\tau \geqq v$ for each τ and $t>\tau$, so the limit will assume positive values, contradicting the assumption that $\mathfrak{D}$ is a domain of uniqueness.

Throughout the paper, we use the following notation:

$$x = (x_1, \cdots, x_n), \qquad r^2 = \sum x_i^2, \qquad u_i = \frac{\partial u}{\partial x_i}, \qquad u_t = \frac{\partial u}{\partial t},$$

and follow the summation convention. For dimensional reasons we shall prefer to write the basic operator in the following form, rather than (1):

$$Lu \equiv a_{ij}u_{ij} + (b_i/r)u_i - u_t, \tag{2}$$

where a_{ij} and b_i are functions of x and t. We also define

$$A(x, t) \equiv a_{ij}x_ix_j/r^2,$$

$$B(x, t) \equiv a_{ii}(x, t) - b_i(x, t)x_i/r.$$

We assume that a_{ij} and b_i/r are locally bounded.

2. The uniformly parabolic case.

THEOREM 1. *Let L satisfy $a_{ij}\xi_i\xi_j \geqq \mu_0|\xi|^2$, where $\mu_0>0$, and $b_ix_i\geqq 0$. Then $\mathfrak{D}$ is a domain of uniqueness for L if*

$$r^2 \leqq k|t| \log\log|t| \tag{3}$$

for some $k<\mu_0$ is satisfied for all points (x, t) in $\mathfrak{D}$ for which $(-t)$ is large enough.

REMARK. The condition $b_ix_i\geqq 0$ could be replaced by the inequality $A(x, t)-B(x, t)\leqq 0$. The latter is implied by the former, since for any positive definite quadratic form, $a_{ij}\xi_i\xi_j/|\xi|^2\leqq a_{ii}$.

Theorem 1 is a corollary of Theorem 3 (in the latter set $D=C=0$), hence will be given no separate proof.

THEOREM 2. *Let L be a parabolic operator with Hölder continuous coefficients and with $A(x, t)$ and $B(x, t)/A(x, t)$ bounded from above. Then no domain described by an inequality*

(4) $$r^2 \leqq k|t| \log|t|$$

($|t|$ large enough) for any $k>0$ is a domain of uniqueness.

PROOF. Let

$$u(x, t) = 1 - |t|^{-\beta}\varphi(r^2/\alpha|t|) \qquad (t < 0),$$

where $\varphi(\eta)$ is the confluent hypergeometric function $\varphi(\eta) = \Phi(a, l; \eta)$ (see [1] for notation), and α, β, a, and c are positive constants to be chosen later. Applying the operator L, we find

$$\begin{aligned} Lu &= A[u_{rr} - (1/r)u_r] + B(1/r)u_r - u_t \\ &= -\frac{4A}{\alpha}|t|^{-\beta-1}\left\{\eta\varphi'' + \left(\frac{B}{2A} - \frac{\alpha}{4A}\eta\right)\varphi' - \frac{\beta\alpha}{4A}\varphi\right\}. \end{aligned}$$

Known properties of φ (see [1]) are

(i) $\varphi, \varphi', \varphi'' \geqq 0$.
(ii) $\varphi(0) = 1$.
(iii) $\varphi(\eta) = (\Gamma(c)/\Gamma(a))e^{\eta}\eta^{a-c}(1+O(\eta^{-1}))$, $\eta \to \infty$.
(iv) $\eta\varphi'' + (c-\eta)\varphi' - a\varphi \equiv 0$.

Subtracting a multiple of (iv) from our expression for Lu, we find

$$Lu = -\frac{4A}{\alpha}|t|^{-\beta-1}\left\{\left[\left(\frac{B}{2A} - c\right) - \left(\frac{\alpha}{4A} - 1\right)\right]\varphi' - \left(\frac{\beta\alpha}{4A} - a\right)\varphi\right\}.$$

We now choose $c \geqq \sup B/2A$ (and positive), $\alpha \geqq \sup 4A$, and $a = \beta$. Thus using (i), we obtain that $Lu \geqq 0$. We now investigate the domain $\mathfrak{D}$ on which $u > 0$. Clearly $u = 0$ on a manifold described by $\varphi(\eta) = |t|^{\beta}$. Taking logarithms and using (iii), we find that for large $|t|$,

$$\eta(1 + o(1)) = \beta \log t,$$

or

$$r^2 = \alpha\beta|t| \log|t| (1 + o(1)).$$

Although α has already been fixed, we are free to choose β as small as desired. Thus $\mathfrak{D}$ can be taken to lie within any domain of the form (4). Since u is bounded by 1, we have that $\mathfrak{D}$ is not a domain of uniqueness. Since Assumption A is satisfied for every domain of the form (4), such domains are likewise not domains of uniqueness.

It may be noted that in case A and B are constant, the above construction yields an exact solution of $Lu = 0$.

3. **The general case.** We introduce the further notation

$$R_0^2(t) = |t| \log\log |t| \qquad (t < 0),$$
$$C(x, t) = \max[0, (A - B)/r],^2$$
$$D(x, t, \tau) = C(x, \tau) |t| / R_0(t),$$

and let $k(t)$ be a function such that $k(t) \leqq 1$ and

$$k(t) \leqq \inf_{t \leqq \tau \leqq 0} \frac{4A^2(x, \tau)}{(D(x, t, \tau) + (D^2 + 4A)^{1/2})^2}.$$

Note that when $b_i x_i \geqq 0$, $C \equiv D \equiv 0$ and $k(t) \leqq \inf A(x, \tau)$.

The following theorem gives a condition for uniqueness in the case $k(t)R_0^2(t)$ is a nonincreasing function of t (that is, a nondecreasing function of $|t|$, since $t \to -\infty$). An analogous condition could be given when kR_0^2 is nondecreasing (this case can always be brought about artificially by choosing k to decrease rapidly enough as $t \to -\infty$).

THEOREM 3. *Let L be parabolic and let $k(t)R_0^2(t)$ be nonincreasing. Let $\mathfrak{D}$ be a domain in which every point with large enough $|t|$ satisfies*

$$r^2 \leqq k(t) |t| \log\log |t| (1 - \epsilon) \tag{5}$$

for some $\epsilon > 0$. Then $\mathfrak{D}$ is a domain of uniqueness.

The proof uses the following

LEMMA. *Let L be parabolic, and $u(x, t)$ a subsolution in a domain $\mathfrak{D}_1$, which is contained in a cylinder $r \leqq R$, $0 \leqq t < \tau$. Assume u is continuous in $\overline{\mathfrak{D}}_1$ and is nonpositive on $\partial\mathfrak{D}_1 \cap \{0 < t < \tau\}$. Then*

$$u(x, \tau) \leqq (1 - \exp[-\alpha R^2/\tau]) \max u(x, 0), \tag{6}$$

where

$$\alpha = \left[\inf_{\mathfrak{D}_1}\left(\frac{A}{1 + \tau C/R}\right)\right]^{-1}.$$

PROOF. It is sufficient to give the proof for the case $R = 1$, for if $R \neq 1$, set $\xi = x/R$, $s = t/R^2$, $\rho = |\xi| = r/R$, $\bar{a}_{ij}(\xi, s) = a_{ij}(x, t)$, $\bar{b}_i = b_i(x, t)$, $\bar{u}(\xi, s) = u(x, t)$. Then

$$\bar{a}_{ij} \frac{\partial^2 \bar{u}}{\partial \xi_i \partial \xi_j} + \frac{\bar{b}_i}{\rho} \frac{\partial \bar{u}}{\partial \xi_i} - \frac{\partial \bar{u}}{\partial s} \geqq 0,$$

in the domain $\rho \leqq 1$, $0 < s < \tau/R^2$. Hence by the result for $R = 1$, $\bar{u}(\xi, s) \leqq (1 - \exp[-\bar{\alpha}/s])\mathrm{Max}\, u(x, 0)$, where

[2] C is bounded at $r = 0$ due to local boundedness of b_i/r and remark following Theorem 1.

$$\bar{\alpha} = \left[\inf \frac{\overline{A}}{1 + s\overline{C}} \right]^{-1},$$

$\overline{A}(\xi, s) = \bar{a}_{ij}\xi_i\xi_j/\rho^2 = A(x, t)$, and $\overline{C}(\xi, s) = \text{Max}[0, (\overline{A} - \overline{B})/\rho] = RC(x, t)$. Thus

$$\bar{\alpha} = \left[\inf_{\mathfrak{D}_1} \frac{A}{1 + tC/R} \right]^{-1} = \alpha,$$

and (6) follows.

So now let $R = 1$ and set $v(x, t) = 1 - \exp[-\alpha(1-r)/t]$. Then

$$Lv = \frac{\alpha}{t} \exp[-\alpha(1 - r)/t] \left\{ \frac{-\alpha A}{t} + \frac{A - B}{r} + \frac{1 - r}{t} \right\}.$$

The quantity in braces is bounded above by

$$((-\alpha A + 1)/t) + C = (1/t)[-\alpha A + 1 + tC],$$

which is $\leqq 0$ provided $\alpha \geqq (1 + tC)/A$. But this is true from the definition of α, provided $0 \leqq t \leqq \tau$. Thus $Lv \leqq 0$. Also notice $v = 1$ for $t = 0$, $v = 0$ for $r = 1$, and $v > 0$ elsewhere. Let $w(x, t) = Mv(x, t) - u$, where $M = \text{Max}\, u(x, 0)$. The lemma is true immediately if $M \leqq 0$, so we assume $M > 0$. Clearly $w(x, 0) \geqq 0$, $w \geqq 0$ on $\partial\mathfrak{D}_1 \cap \{0 < t < \tau\}$, and $(-w)$ is a subsolution except at $r = 0$, where v is irregular. Suppose w assumed negative values. Then the minimum of w in $\overline{\mathfrak{D}}_1$ could not be assumed on $\partial\mathfrak{D}_1 \cap \{0 \leqq t \leqq \tau\}$. Furthermore, by the maximum principle it could not be assumed at any interior point at which w is regular, nor for $t = \tau$, except possibly at $r = 0$. This leaves only the possibility of a minimum at $r = 0$. But this is likewise excluded, because it is clear from the definition of v that w_{x_1}, as a function of x_1, suffers a negative jump discontinuity at $r = 0$, which cannot occur at a minimum. Thus $w \geqq 0$ and

$$u \leqq Mv \leqq M(1 - \exp[-\alpha/t]).$$

Setting $t = \tau$, we obtain the conclusion of the lemma.

Proof of Theorem 3. Define, for $i = 1, 2, \cdots$ and some $\beta > 0$, $t_i = -e^{\beta i}$,

$$\Delta_i t = t_{i-1} - t_i = |t_i| (1 - e^{-\beta}),$$

$$R_i^2 = |t_i| (\log\log |t_i|) k(t_i)(1 - \epsilon) \tag{7}$$

$$= \Delta_i t \log(\beta i) k(t_i)(1 - \epsilon)/(1 - e^{-\beta}).$$

Let $u(x, t)$ be a subsolution in $\mathfrak{D}$ satisfying $u \leqq 0$ on $\partial\mathfrak{D}$, and define

$$M_i = \sup_{(x, t_i) \in \mathfrak{D}} u(x, t_i).$$

Consider the cylinder $\sum_i: r<R_i,\ t_i<t<t_{i-1}$. Since kR_0^2 is nonincreasing, $R_i^2=k(t_i)R_0^2(t_i)(1-\epsilon)$, and $\mathfrak{D}$ lies within $r\leqq kR_0^2(1-\epsilon)$, we know that $\mathfrak{D}\cap\{t_i<t<t_{i-1}\}$ is contained in $\sum_i$. Hence by the lemma,

$$M_{i-1} \leqq (1-\exp[-\alpha_i R_i^2/\Delta_i t])M_i, \tag{8}$$

where

$$\alpha_i = \inf_{\Sigma_i}\left[\frac{A}{1+\Delta_i tC/R_i}\right]^{-1}.$$

However, we know that

$$\frac{\Delta_i t}{R_i} = \frac{|t_i|(1-e^{-\beta})}{R_0(t_i)(k(t_i))^{1/2}(1-\epsilon)^{1/2}}.$$

Thus

$$\alpha_i \leqq \left[\inf_{G(t_i)}\left(\frac{A}{1+\gamma|t_i|C(x,t)/R_0(t_i)(k(t_i))^{1/2}}\right)\right]^{-1},$$

where $G(\tau)=\{r<R_0(\tau);\ \tau<t<0\}$ and $\gamma=(1-e^{-\beta})(1-\epsilon)^{-1/2}$. This is because $\sum_i$ is contained within $G(t_i)$. Next we recall that for all $(x,t)\in G(\tau)$,

$$(k(\tau))^{1/2} \leqq \frac{2A(x,t)}{D+(D^2+4A)^{1/2}}.$$

Thus

$$(k(\tau))^{1/2} \leqq ((D^2+4A^{1/2}-D))/2, \quad \text{and} \quad k+Dk^{1/2}\leqq A,$$

or

$$k(\tau) \leqq A/(1+D/k^{1/2}).$$

We set $\tau=t_i$ in this inequality, and take the infimum over $G(t_i)$. We thus obtain

$$k(t_i) \leqq 1/\alpha_i, \tag{9}$$

provided β is chosen large enough so that $\gamma\geqq 1$. It follows from (7) and (9) that

$$\begin{aligned}\alpha_i R_i^2/\Delta_i t &\leqq (k(t_i))^{-1}\log(\beta i)k(t_i)(1-\epsilon)/(1-e^{-\beta})\\ &= \log(\beta i)(1-\epsilon)^{1/2}\gamma^{-1} \leqq \log(\beta i).\end{aligned}$$

Hence from (8),

$$M_{i-1} \leqq (1-(\beta i)^{-1})M_i.$$

From this it follows that

$$M_i > \prod_{j=m+1}^{i} (1 - (\beta j)^{-1})^{-1} M_m \qquad (i > m).$$

However, the product $\prod^\infty (1-(\beta j)^{-1})^{-1}$ is divergent; hence if $M_m > 0$ for any m, u will be unbounded. Thus u is either nonpositive or unbounded. This shows that $\mathfrak{D}$ is a domain of uniqueness and the theorem is proved.

We conclude with a maximum principle for cylindrical domains.

THEOREM 4. *Let the parabolicity of L be uniform in t. Also assume that $(A-B)/r$ is bounded from above uniformly in t. Then any cylindrical domain with finite cross-section is a domain of uniqueness.*

PROOF. Let $u(x, t)$ be a subsolution nonpositive on the boundary, and define $M_i = \sup_x u(x, -i)$. The uniformity assumptions and the lemma imply that there is a $\kappa < 1$ such that $M_{i-1} \leqq \kappa M_i$ for all i. Thus the M_i either all vanish or grow exponentially in i, and the theorem is proved.

The uniformity assumptions could of course be considerably weakened. However, we mention the example

$$\frac{1}{1+t^2} u_{xx} - u_t = 0,$$

which has the solution $u(x, t) = \sin x \exp(-\tan^{-1} t)$ vanishing on the boundary of the strip $0 < x < \pi$. The strip is therefore not a domain of uniqueness for this operator.

BIBLIOGRAPHY

1. A. Erdélyi et al., *Higher transcendental functions*, Vol. 1, McGraw-Hill, New York, 1953.

2. P. Fife, *Solutions of parabolic boundary problems existing for all time*, Arch. Rational Mech. Anal. 16 (1964), 155–186.

3. ——— *A note on exterior Dirichlet problems and an application to boundary layer theory*, J. Differential Equations 1 (1965), 95–113.

4. N. Meyers and J. Serrin, *The exterior Dirichlet problem for second order elliptic partial differential equations*, J. Math. Mech. 9 (1960), 513–538.

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