TENSOR PRODUCTS OF F-VECTOR SPACES[1]

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Kuranishi [1] has defined the concept of an F-vector space. Each F-vector space determines a pair of numbers (m, p) called the characteristic. The characteristic constitutes a generalization to this class of in general infinite dimensional vector spaces of the notion of dimension for finite dimensional vector spaces. The latter, in fact, correspond to F-vector spaces of characteristic $(m, 0)$ with m equaling the dimension. Isomorphism of F-vector spaces, over the same field of course, is such that they are isomorphic if and only if they have the same characteristic.

Except for a very special case, the construction of the space of formal curves, Kuranishi left open the question of forming tensor products of F-vector spaces. It will be shown that *the tensor product of two F-vector spaces having characteristics (m, p) and (m', p') respectively exists and is an F-vector space of characteristic $(mm', p+p')$*. It is immediately seen that this generalizes the well-known result for finite dimensional vector spaces.

All vector spaces are taken over some infinite field.

A PF-vector space $H=(H, H^{(l)}, B^{(l)}, h^{(l)})$ is a vector space such that

(1) $H^{(l)}$ and $B^{(l)}$ are vector subspaces for each nonnegative integer l;

(2) $H=H^{(0)}\supseteq H^{(1)}\supseteq \cdots \supseteq H^{(l)}\supseteq H^{(l+1)}\supseteq \cdots$;

(3) $\bigcap_l H^{(l)}=\{0\}$;

(4) $H^{(l)}=B^{(l)}\oplus H^{(l+1)}$;

(5) $B^{(l)}$ has finite dimension d^l and $h^{(l)}=\{h_1^l, h_2^l, \cdots, h_{d^l}^l\}$ is an ordered basis for $B^{(l)}$.

An F-vector space $H=(H, H^{(l)}, B^{(l)}, h^{(l)})$ is a PF-vector space such that there are integers p and k and a real number m_1 such that for all sufficiently large l

$$m_1(l-k)^p \leqq \dim(H/H^{(l)}) \leqq m_1(l+k)^p.$$

p is called the degree and m defined to be equal to $p!m_1$ is called the

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multiplicity. m and p are uniquely determined and (m, p) is the *characteristic* mentioned above.

Let $H=(H, H^{(l)}, B^{(l)}, h^{(l)})$ and $H'=(H', H'^{(l)}, B'^{(l)}, h'^{(l)})$ be PF-vector spaces. The vector space $H^*=H\otimes H'$ is called the *tensor product of H and H′* and consists of all formal infinite linear combinations of the linearly independent elements $x\otimes y$ where $x\in h^{(l)}$ and $y\in h'^{(j)}$ for some l and j.[3] One distinguishes the following subspaces

$$B^{*(j)} = \sum_{s=0}^{j} \oplus \left(B^{(j-s)} \otimes B'^{(s)}\right), \qquad j \geqq 0,$$

$$H^{*(j)} = \sum_{s=0}^{j} \oplus \left(H^{(j-s)} \otimes H'^{(s)}\right), \qquad j \geqq 0.$$

An ordered basis $h^{*(j)}$ of $B^{*(j)}$ consists of the elements $h_q^{j-s}\otimes h_r'^s$, $0\leqq s\leqq j$, $1\leqq q\leqq d^{j-s}$, $1\leqq r\leqq d^s$ ordered as follows:

$$h_q^{j-s} \otimes h_r'^s < h_u^{j-t} \otimes h_v'^t$$

if and only if $s<t$, or $s=t$ and $r<v$, or $s=t$ and $r=v$ and $q<u$.

Taking into account the various natural isomorphisms, it is readily verified that $H^*=(H^*, H^{*(l)}, B^{*(l)}, h^{*(l)})$ is a PF-vector space. If H and H' are also F-vector spaces one wants to show that H^* is also an F-vector space and to determine its characteristic. Although the condition for being an F-vector space is fairly stringent it does permit enough malignancy to have defeated every effort at the direct construction of bounds for $H^*/H^{*(l)}$ in terms of those for $H/H^{(l)}$ and $H'/H'^{(l)}$ attempted by the author. It appears that passing to isomorphic F-vector spaces on which suitable computations can be made is the only means at hand.

Let H and H'' be PF-vector spaces. A linear mapping F from H into H'' is called *analytic* if and only if there is an integer k such that $F(H^{(l)})\subseteq H''^{(l-k)}$ for all sufficiently large l. H and H'' are *isomorphic* if and only if there are analytic linear mappings F of H into H'' and G of H'' into H such that $G\circ F$ and $F\circ G$ are identity mappings in their respective domains.

Suppose that H and H'' are isomorphic PF-vector spaces and that H is also an F-vector space of characteristic (m, p). Then the mappings F and G are bijections which are used to identify H and H''. Moreover, there are integers k and k' such that

[3] More precisely, H^* is the vector space generated by $B^{*(l)}$ and $h^{*(l)}$, which are defined just below, as constructed by Kuranishi [1, pp. 230–231] but omitting the condition for an F-vector space.

$$H^{(l)} \subseteq H''^{(l-k)} \subseteq H^{(l-k-k')}.$$

But then, for some k_1 and all sufficiently large l,

$$\frac{m}{p!}(l+k_1)^p \geqq \dim(H/H^{(l)}) \geqq \dim(H''/H''^{(l-k)})$$

$$\geqq \dim(H/H^{(l-k-k')}) \geqq \frac{m}{p!}(l-k_1)^p.$$

Therefore, H'' is an F-vector space of characteristic (m, p). Thus, *if $H=(H, H^{(l)}, B^{(l)}, h^{(l)})$ is an F-vector space, $H''=(H'', H''^{(l)}, B''^{(l)}, h''^{(l)})$ is a PF-vector space, and H and H'' are isomorphic, then H'' is an F-vector space having the same characteristic as H.* Kuranishi[4] also proved that *two F-vector spaces having the same characteristic are isomorphic.*

Let H, H', H'', H''' be PF-vector spaces. Suppose that H is isomorphic to H'' under the analytic linear mappings F, G and that H' is isomorphic to H''' under the analytic linear mappings F', G'. Form $H^*=H\otimes H'$ and $H^{**}=H''\otimes H'''$. Define F^* and G^* as follows: $F^*(x\otimes y)=F(x)\otimes F'(y)$ for $x\in H$ and $y\in H'$; $G^*(x\otimes y)=G(x)\otimes G'(y)$ for $x\in H''$ and $y\in H'''$. Then one easily demonstrates that F^* and G^* are analytic linear mappings such that F^*G^* and G^*F^* are identities on their respective domains. Thus, *if H and H'' are isomorphic PF-vector spaces and H' and H''' are isomorphic PF-vector spaces, then the tensor products $H\otimes H'$ and $H''\otimes H'''$ are isomorphic PF-spaces.*

Let $H=(H, H^{(l)}, B^{(l)}, h^{(l)})$ and $H'=(H', H'^{(l)}, B'^{(l)}, h'^{(l)})$ be F-vector spaces of characteristics (m, p) and (m', p') respectively. Let $H^*=H\otimes H'$ be the tensor product of H and H'. The characteristic of H^* will now be determined.

If $p=0$, the $m=m_1$ is equal to the dimension of H. Let $B''^{(0)}$ be a vector space of dimension m and let $h''^{(0)}$ be an ordered basis for $B''^{(0)}$. Let $B''^{(l)}=\{0\}$ and $h''^{(l)}=\varnothing$ for $l>0$. Let H'' be the F-vector space generated by $B''^{(l)}$ and $H''^{(l)}$. Then H and H'' are isomorphic F-vector spaces because they have the same characteristic. Let $H^{**}=H''\otimes H'$. The vector spaces $H^{**}/H^{**(l)}$ and $B''^{(0)}\otimes(B''^{(0)}\oplus\cdots\oplus B''^{(l-1)})$ are isomorphic and hence have the same dimension. Hence,

$$mm_1'(l-k')^{p'} \leqq \dim(H^{**}/H^{**(l)}) \leqq mm_1'(l+k')^{p'}$$

[4] The preceding result is a slight modification of a result of Kuranishi's [1, Proposition I.2, Theorem I.1].

where $m' = p'!m_1'$, for sufficiently large l. This implies that H^{**} has characteristic (mm', p'), and, since H^* and H^{**} are isomorphic PF-vector spaces, H^* has the same characteristic.

Now let p be greater than zero.

It is convenient to set the symbol x^q equal to zero when x is negative and q is positive, and this will be assumed henceforth. Note then that one can choose k so large that

$$\begin{aligned} m_1(n-k)^p &\leqq \dim(B^{(0)} \otimes \cdots \otimes B^{(n-1)}) \leqq m_1(n+k)^p, \\ m_1'(n-k)^{p'} &\leqq \dim(B'^{(0)} \otimes \cdots \otimes B'^{(n-1)}) \leqq m_1'(n+k)^{p'} \end{aligned} \tag{1}$$

for all $n \geqq 0$, and also such that

$$k \geqq 1/(2m_1). \tag{2}$$

Let $B''^{(l)}$ be a vector space of dimension equal to

$$[m_1(l+1+k)^p] - [m_1(l+k)^p]^{5}$$

for $l > 0$. Let $B''^{(0)}$ be a vector space of dimension $[m_1(1+k)^p]$. Let $h''^{(l)}$ be an ordered basis for $B''^{(l)}$ for each $l \geqq 0$. Clearly

$$\dim(B''^{(0)} \otimes \cdots \otimes B''^{(l-1)}) \leqq m_1(l+k)^p. \tag{3}$$

Also, utilizing the inequality (2),

$$\dim(B''^{(0)} \otimes \cdots \otimes B''^{(l-1)}) \geqq m_1(l+k)^p - 1 \geqq m_1(l-k)^p. \tag{4}$$

Let $H'' = (H'', B''^{(l)}, H''^{(l)}, h''^{(l)})$ be the F-vector space generated by $B''^{(l)}$ and $h''^{(l)}$ as constructed by Kuranishi. H'' has characteristic (m, p), $m = p!m_1$, and thus is isomorphic to H.

Note that

$$pm_1(j-k-2)^{p-1} \leqq \dim B''^{(j)} \leqq pm_1(j+k+2)^{p-1}. \tag{5}$$

Let H^{**} be the tensor product $H'' \otimes H'$. Then

$$\begin{aligned} \dim(H^{**}/H^{**(l)}) &= \dim(B^{**(0)} \otimes \cdots \otimes B^{**(l-1)}) \\ &= \sum_{j=0}^{l-1} (\dim B''^{(l-1-j)}) \cdot (\dim(B'^{(0)} \otimes \cdots \otimes B'^{(j)}). \end{aligned}$$

Thus, utilizing inequalities (1) and (5)

$$pm_1m_1' \sum_{j=0}^{l-1} (l-1-j-k-2)(j+1-k)^{p'} \leqq \dim(H^{**}/H^{**(l)}) \tag{6}$$

and

[5] $[y]$ is the greatest integer in y.

$$(7)\quad \dim(H^{**}/H^{**(l)}) \leqq pm_1m_1' \sum_{j-0}^{l-1} (l-1-j+k+2)^{p-1}(j+k+1)^{p'}.$$

Note that

$$(8)\qquad \binom{j}{p} \leqq \frac{1}{p!} j^p \leqq \binom{j+p}{p}$$

and

$$(9)\qquad \sum_{s=0}^{r} \binom{r-s+q}{q}\binom{s+t}{t} = \binom{r+q+t+1}{q+t+1}.$$

The suprabound (7)

$$\begin{aligned}
&pm_1m_1' \sum_{j=0}^{l-1} (l+k+1-j)^{p-1}(j+k+1)^{p'} \\
&\qquad \leqq p!p'!m_1m_1' \sum_{j=0}^{l+2k+2} \frac{1}{(p-1)!} (l+2k+2-j)^{p-1} \frac{1}{p'!} j^{p'} \\
(10)\qquad &\qquad \leqq mm' \sum_{j=0}^{l+2k+2} \binom{l+2k+2-j+p-1}{p-1}\binom{j+p'}{p'} \\
&\qquad = mm' \binom{l+2k+2+p+p'}{p+p'} \\
&\qquad \leqq \frac{mm'}{(p+p')!} (l+k')^{p+p'}
\end{aligned}$$

for some k' and sufficiently large l.

The infrabound (6)

$$\begin{aligned}
&pm_1m_1' \sum_{j=0}^{l-1} (l-3-j-k)^{p-1}(j+1-k)^{p'} \\
&\qquad \geqq mm' \sum_{j=0}^{l-2k-2} \frac{1}{(p-1)!} (l-2k-2-j)^{p-1} \frac{1}{p'!} j^{p'} \\
(11)\qquad &\qquad \geqq mm' \sum_{j=0}^{l-2k-2+1} \binom{l-2k-2-j}{p-1}\binom{j}{p'} \\
&\qquad = mm' \binom{l-2k-2+1}{p+p'} \\
&\qquad \geqq \frac{mm'}{(p+p')!} (l-k'')^{p+p'}
\end{aligned}$$

for some k'' and sufficiently large l.

The bounds (10) and (11) imply that H^{**} has characteristic $(mm', p+p')$. Since H^* and H^{**} are isomorphic PF-vector spaces, H^* has the same characteristic.

References

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