DENSITY OF INTEGER SEQUENCES

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1. Let $N=\{n_k\}$ be an increasing sequence of positive integers. Then N fulfills *condition D* if it contains a sequence of blocks

$$B_\nu = [u_\nu, v_\nu] \cap N, \qquad 1 \leqq u_\nu < v_\nu, \tag{1}$$

for which $v_\nu - u_\nu \to \infty$, and $1+v_\nu-u_\nu \leqq C\cdot|B_\nu|$. Here $|S|$ always denotes the number of elements in a finite set S.

Let $N^1, \cdots, N^r$ be r sequences fulfilling condition D, and for each real number x, let

$$\Delta(x)=\{(n_1x, \cdots, n_rx): n_1<n_2< \cdots <n_r \text{ and } n_s\in N^s\ (1\leqq s\leqq r)\}.$$

Thus $\Delta(x)$ is a denumerable subset of R^r.

THEOREM 1. *For all but a denumerable set of real numbers x, $\Delta(x)$ is dense (modulo 2π) in R^r.*

The statement for $r=1$ is proved, with more precision, by Amice [1] and Kahane [2]. It will be clear from the proof that the inequalities in the definition of $\Delta(x)$ can be strengthened almost arbitrarily.

First we express the exceptional set in Theorem 1 as a denumerable union of closed sets. Let $U_1, \cdots, U_j, \cdots$ be a sequence of open sets in R^r forming a base for the topology, and Λ the subgroup of R^r of integral vectors. Put, for each j,

$$E_j = \{x \in R: \Delta(x) \cap (U_j + 2\pi\Lambda) = \varnothing\}.$$

Then each E_j is closed and we must show that each is denumerable. In the opposite case some E_j would contain a compact nondenumerable subset; hence a homeomorph of the Cantor set, and so E_j would carry a continuous probability measure μ, ($\mu(D)=0$ for every denumerable set D). We shall now state a theorem on probability distributions that implies Theorem 1.

2. Let M be the set of r-tuples $(n_1, \cdots, n_r)\in N^1\times \cdots \times N^r$ defined by the inequalities $n_1<n_2< \cdots <n_r$ and X a real random variable whose distribution is continuous.

THEOREM 2. *For a certain sequence $(n_{i1}, \cdots, n_{ir})\in M$, the r-dimensional variables $Y_i=(n_{i1}X, \cdots, n_{ir}X)$, $i=1, 2, 3, \cdots$, tend modulo 2π to uniform distribution.*

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DEDUCTION OF THEOREM 1. We suppose, after our remarks on Theorem 1, that for some open set U (one of the U_j, for example)

$$P\{\Delta(X) \cap (U + 2\pi\Lambda) = \emptyset\} = 1.$$

But then $P\{Y_i \in U+2\pi\Lambda\} = 0$, $i=1, 2, 3, \cdots$, contradicting the fact that $U+2\pi\Lambda$ has positive measure in the quotient group $R^r/2\pi\Lambda$. This completes the deduction of Theorem 1.

3. **Proof of Theorem 2.** By Weyl's criterion, a sequence Y_i tends to uniform distribution (modulo 2π) if the r-dimensional characteristic functions, ψ_i, of the Y_i converge to 0 at each element of Λ except 0. For the variables Y_i, the characteristic functions ψ_i are determined by the characteristic function ϕ of the real variable X:

$$\psi_i(m_1, \cdots, m_r) = \phi(m_1 n_{i1} + \cdots + m_r n_{ir}).$$

A theorem of Wiener [**3**, p. 221] states that

$$\lim_{m\to\infty} \frac{1}{m+1} \sum_0^m |\phi(j)| = 0,$$

and in fact the proof cited shows that

$$\lim_{m\to\infty} \frac{1}{m+1} \sum_p^{p+m} |\phi(j)| = 0,$$

uniformly for all integers p.

For any finite set H of elements $\neq 0$ of Λ, and any $\delta > 0$, we shall find a finite subset S of M so that

$$\sum_S |\phi(m_1 n_1 + \cdots + m_r n_r)| < \delta |S| \tag{2}$$

for each $(m_1, \cdots, m_r) \in H$. Then, for *at least one* $(n_1^*, \cdots, n_r^*) \in S$, and for *every* $(m_1, \cdots, m_r) \in H$,

$$|\phi(m_1 n_1^* + \cdots + m_r n_r^*)| < \delta |H|;$$

this easily implies the existence of the asserted sequence $(n_{i1}, \cdots, n_{ir})$ in M. The proof will show that S can be chosen so that (2) holds uniformly for any $(m_1, \cdots, m_r)$ containing at least one coefficient $m_j \neq 0$ but $|m_j| \leqq C'$.

Given a number $b > 1$, choose, using (1), intervals $[u_1, v_1], \cdots, [u_r, v_r]$ such that

$$b \leqq v_1 - u_1, \qquad b v_s \leqq v_{s+1} - u_{s+1}, \qquad 1 \leqq s < r,$$

and

$$1 + v_s - u_s \leqq C \cdot | N^s \cap [u_s, v_s] |, \qquad 1 \leqq s \leqq r.$$

Hence

$$\prod_{s=1}^{r} (v_s - u_s + 1) \leqq C^r \prod_{s=1}^{r} | N^s \cap [u_s, v_s] | .$$

Let Q denote the rectangle in R^r $[u_1, v_1] \times \cdots \times [u_r, v_r]$, so that the last inequality is just

$$| Q \cap \Lambda | = \prod_{s=1}^{r} (v_s - u_s + 1) \leqq C^r | Q \cap N^1 \times \cdots \times N^r | .$$

Moreover, the inequalities $n_1 < \cdots < n_r$ are satisfied by all the elements of $Q \cap \Lambda$ except at most $rb^{-1} | Q \cap \Lambda |$. Thus if b is sufficiently large

$$| Q \cap \Lambda | \leqq 2C^r | Q \cap M | .$$

It is enough to attain, therefore,

$$\sum_{Q \cap \Lambda} | \phi(m_1 n_1 + \cdots + m_r n_r) | \leqq \frac{1}{2} \delta C^{-r} | Q \cap \Lambda | . \tag{3}$$

Suppose, for example, that $1 \leqq m_r \leqq C'$. By holding $n_1, \cdots, n_{r-1}$ fixed, and varying n_r in $[u_r, v_r]$, we obtain from the sums $m_1 n_1 + \cdots + m_r n_r$ an arithmetic progression of at least b terms, and difference at most C'. So

$$\sum_{u_r \leqq n_r \leqq v_r} | \phi(m_1 n_1 + \cdots + m_r n_r) | \leqq \sup_p \sum_{p}^{p+h} | \phi(j) | ,$$

where $b \leqq v_r - u_{r+1} \leqq h \leqq C'(v_r - u_r + 1)$. Hence

$$\sup_p \sum_{p+1}^{p+h} | \phi(j) | = o(h) = o(v_r - u_r + 1) \quad \text{as } b \to \infty .$$

These estimates are uniform with respect to $n_1, \cdots, n_{r-1}$ and so (3) holds if b is sufficiently large. Theorem 2 is completely proved.

References

1. Y. Amice, *Un théorème de finitude*, Ann. Inst. Fourier **14** (1964), 527–531.

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3. A. Zygmund, *Trigonometric series*, Monogr. Mat., Bd. 5, PWN, Warsaw, 1935.

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