REMARK ON THE PURITY OF THE BRANCH LOCUS

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The purpose of the following note is to give a simple proof for the purity of the branch locus for local rings of characteristic $p>0$.

Let R be a regular local ring of characteristic $p>0$, K its quotient field, k its residue class field. Let L be a finite separable extension of K and S be the integral closure of R in L. It is well known that S is a finitely generated R-module and that $S=\bigcap_{h(\mathfrak{P})=1} S_{\mathfrak{P}}$, where $\mathfrak{P}$ runs over all prime ideals of S of height $h(\mathfrak{P})=1$.

LEMMA 1. *R is a faithfully flat R^p-module.*

This follows from the commutative diagram

$$\begin{array}{ccll} R & \rightarrow & \hat{R} & = k[[x_1, \cdots, x_n]] \\ \uparrow & & \uparrow & \\ R^p & \rightarrow & \hat{R}^p & = k^p[[x_1^p, \cdots, x_n^p]], \end{array}$$

where $\hat{R}$ is the completion of R, $\hat{R}=k[[x_1, \cdots x_n]]$, a representation of $\hat{R}$ as a ring of formal power series over k, and from the fact that the extensions $R\rightarrow\hat{R}$, $R^p\rightarrow\hat{R}^p$ and $k^p[[x_1^p, \cdots x_n^p]]\rightarrow k[[x_1, \cdots x_n]]$ are faithfully flat.

REMARK. If R is a commutative ring, p an integer, T a multiplicatively closed subset of R, then $R_T=R_{T^p}$.

LEMMA 2. $R[S^p]\cong R\otimes_{R^p} S^p$.

Consider the sequence

$$R\otimes_{R^p} S^p \rightarrow R\otimes_{R^p} L^p \rightarrow K\otimes_{K^p} L^p \rightarrow K[L^p] \subseteq L.$$

The first arrow is an injection by Lemma 1, the second is an isomorphism, as $R\otimes_{R^p} L^p\cong(R\otimes_{R^p} K^p)\otimes_{K^p} L^p$ and $K\cong R\otimes_{R^p} K^p$ by the remark. The third arrow is an isomorphism, as L is separable over K. As $R[S^p]$ is the image of $R\otimes_{R^p} S^p$ in $K[L^p]$, the lemma follows.

LEMMA 3. $R[S^p]=\bigcap_{h(\mathfrak{q})=1} R[S^p]_{\mathfrak{q}}$, *where $\mathfrak{q}$ runs over all prime ideals of $R[S^p]$ of height* 1.

For each $\mathfrak{q}$ there is exactly one prime ideal $\mathfrak{P}$ of S with $\mathfrak{P}\cap R[S^p]=\mathfrak{q}$, namely

$$\mathfrak{P} = \sqrt{\mathfrak{q}} = \{x\in S \mid x^p \in \mathfrak{q}\},$$

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and $h(\mathfrak{P})=h(\mathfrak{q})$. By the remark and by Lemma 2 we have $R[S^p]_{\mathfrak{q}} \cong R\otimes_{R^p}(S_{\mathfrak{P}})^p$. Let

$$y \in \bigcap_{h(\mathfrak{P})=1} R \otimes_{R^p} (S_{\mathfrak{P}})^p \subseteq R \otimes_{R^p} L^p.$$

Then

$$y = \sum_{i=1}^{m} r_i \otimes l_i \qquad (r_i \in R,\ l_i \in L^p).$$

Since S^p is an integrally closed noetherian domain, the l_i $(i=1 \cdots m)$ are contained in almost all of the $(S_{\mathfrak{P}})^p$. Let $\mathfrak{P}_1, \cdots, \mathfrak{P}_r$ be the exceptional prime ideals. Then

$$y \in R \otimes_{R^p}\left(\bigcap_{\mathfrak{P}\neq\mathfrak{P}_i} (S_{\mathfrak{P}})^p\right) \cap \bigcap_{i=1}^{r} R \otimes_{R^p} (S_{\mathfrak{P}_i})^p.$$

Since R is flat over R^p the tensor product commutes with finite intersections (Bourbaki, *Éléments de mathématique. Algèbre commutative*, Hermann, Paris, Chapter I, §2, Proposition 6). Thus

$$y \in R \otimes_{R^p}\left(\bigcap_{h(\mathfrak{P})=1} (S_{\mathfrak{P}})^p\right) = R \otimes_{R^p} S^p.$$

LEMMA 4. *Let* $\mathfrak{P}\in\operatorname{Spec}(S)$, $\mathfrak{P}\cap R[S^p]=\mathfrak{q}$. $S_{\mathfrak{P}}$ *is unramified over* R *if and only if* $S_{\mathfrak{P}}=R[S^p]_{\mathfrak{q}}$.

Let $\mathfrak{P}\cap R=\mathfrak{g}$ and $k_{\mathfrak{P}}$, $k_{\mathfrak{q}}$, $k_{\mathfrak{g}}$ be the residue class fields of $S_{\mathfrak{P}}$, $R[S^p]_{\mathfrak{q}}$ and $R_{\mathfrak{g}}$ respectively. Then $k_{\mathfrak{q}}=k_{\mathfrak{g}}[k_{\mathfrak{P}}^p]$.

If $S_{\mathfrak{P}}$ is unramified over R, then it is also unramified over $R[S^p]$. It follows that $k_{\mathfrak{P}}=k_{\mathfrak{q}}=k_{\mathfrak{g}}[k_{\mathfrak{P}}^p]$, as $k_{\mathfrak{P}}$ is at the same time separable and purely inseparable over $k_{\mathfrak{q}}$. $S_{\mathfrak{P}}$ is a finitely generated $R[S^p]_{\mathfrak{q}}$-module by the above remark and $S_{\mathfrak{P}}=R[S^p]_{\mathfrak{q}}+\mathfrak{q}S_{\mathfrak{P}}$. By the lemma of Nakayama, we have $S_{\mathfrak{P}}=R[S^p]_{\mathfrak{q}}$.

Conversely, let $S_{\mathfrak{P}}=R[S^p]_{\mathfrak{q}}$. Then $k_{\mathfrak{P}}=k_{\mathfrak{q}}=k_{\mathfrak{g}}[k_{\mathfrak{P}}^p]$. Since $k_{\mathfrak{P}}$ is finite over $k_{\mathfrak{g}}$ it follows that $k_{\mathfrak{P}}$ is separable over $k_{\mathfrak{g}}$ and $k_{\mathfrak{P}}\cong k_{\mathfrak{g}}\otimes_{(k_{\mathfrak{g}})^p}(k_{\mathfrak{P}})^p$. The ideal $i=\mathfrak{g}R_{\mathfrak{g}}\otimes(S_{\mathfrak{P}})^p+R_{\mathfrak{g}}\otimes(\mathfrak{P}S_{\mathfrak{P}})^p$ of

$$R_{\mathfrak{g}} \otimes_{(R_{\mathfrak{g}})^p} (S_{\mathfrak{P}})^p \cong R[S^p]_{\mathfrak{g}} = S_{\mathfrak{P}}$$

is the maximal ideal of this ring, as $R_{\mathfrak{g}}\otimes_{(R_{\mathfrak{g}})^p}(S_{\mathfrak{P}})^p/i\cong k_{\mathfrak{g}}\otimes_{(k_{\mathfrak{g}})^p}(k_{\mathfrak{P}})^p$ is a field, thus $\mathfrak{P}S_{\mathfrak{P}}=\mathfrak{g}S_{\mathfrak{P}}+\mathfrak{P}^pS_{\mathfrak{P}}$. From the lemma of Nakayama, we have $\mathfrak{P}S_{\mathfrak{P}}=\mathfrak{g}S_{\mathfrak{P}}$ and $S_{\mathfrak{P}}$ is unramified over R.

THEOREM. *Let* $S_{\mathfrak{P}}$ *be unramified over* R *for all prime ideals* $\mathfrak{P}$ *of* S

with height 1. *Then S is unramified over R.*[1]

Let $\mathfrak{P} \in \operatorname{Spec}(S)$ be a prime ideal of height 1 and $\mathfrak{q} = \mathfrak{P} \cap R[S^p]$. Then by Lemma 4 we have $S_{\mathfrak{P}} = R[S^p]_{\mathfrak{q}}$. Lemma 3 gives us $S = \bigcap_{h(\mathfrak{P})=1} S_{\mathfrak{P}} = \bigcap_{h(\mathfrak{q})=1} R[S^p]_{\mathfrak{q}} = R[S^p]$. Now by Lemma 4, S is unramified over R.

It is easy to extend Lemma 4 to the following criterion: Let $R \subseteq S$ be noetherian local rings with maximal ideals $m \subseteq M$ and residue class fields $k \subseteq K$. If char $k = p > 0$, if K is finite over k and S is a finitely generated $R[S^p]$-module, then S is unramified over R if and only if $S = R[S^p]$.

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[1] ADDED IN PROOF. It is sufficient to assume that R is a local domain, which is flat over R^p. However, in a forthcoming paper (Amer. J. Math.) the author will show that R is then a regular local ring.