A NONLINEAR STEADY STATE TEMPERATURE PROBLEM FOR A SEMI-INFINITE SLAB

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We propose to investigate the following boundary value problem:

(1) $w_{xx}+w_{yy}=0,\ 0<x<\pi,\ 0<y<\infty$,

(2) $w(0,\ y)=f_1(y),\ 0\leqq y<\infty$,

(3) $w(\pi,\ y)=f_2(y),\ 0\leqq y<\infty$,

(4) $-w_y(x,\ 0)=G[x;\ w(x,\ 0)],\ 0<x<\pi$,

where f_1, f_2 are real-valued continuous functions on $[0,\ \infty[$ vanishing at infinity, and G is a nonlinear real-valued function of two real variables satisfying the following conditions:

$G[x;\ u]$ is strictly decreasing in u, $G[x;\ c]=0$ where c is a constant such that

(5) $c\geqq \sup\ \sup_x(f_1(x),\ f_2(x))$,

(6) $G[x;\ u]$ is continuous in x and u jointly.

Physically, w is the temperature, and the condition (4) with G satisfying (5) and (6) is a generalization of Newton's law of cooling; $-w_y(x,\ 0)=l(c_0-w(x,\ 0))$, l and c_0 being constants.

Mann and Blackburn [1] investigated a particular case of the foregoing problem corresponding to $f_1=0=f_2$, $c=1$, and G independent of the first argument x. The reason we consider this problem again is two-fold: first, the argument given in [1] in the proof of the existence theorem (loc. cit. Theorem 3) does not directly carry over to the present more general case, and second, we wish to present a method for constructing the solution in certain particular cases, whereas the authors in [1] were more concerned with existence theorems. Our main tool is a fixed point theorem of H. Schaefer [2].

By a solution of the boundary value problem (1)–(6), we mean a function w harmonic in the open strip, satisfying (2)–(6), continuous in $0\leqq x\leqq \pi$, $0\leqq y<\infty$, and vanishing at infinity. We postulate w to be of the form

(7) $w=w_1+w_2$, where w_1, w_2 are harmonic in the open strip, and

(8) $w_1(0,\ y)=w_1(\pi,\ y)=0,\ 0\leqq y<\infty$,

(9) $-w_{1,y}(x,\ 0)=G[x;\ w(x;\ 0)],\ 0<x<\pi$,

(10) $w_2(0,\ y)=f_1(y)$,

(11) $w_2(\pi,\ y)=f_2(y)$,

(12) $w_{2,y}(x,\ 0)=0,\ 0<x<\pi$.

By proceeding as in [1], one finds

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$$w_1(x, y) = \frac{1}{\pi}\int_0^\pi \log \frac{1 - 2\exp(-y)\cos(x - z) + \exp(-y)}{1 - 2\exp(-y)\cos(x + z) + \exp(-y)} \cdot G[z; w(z; 0)]dz, \qquad 0 < x < \pi,\ 0 < y < \infty. \tag{13}$$

We now seek an integral representation formula for w_2. We use the technique of Fourier transforms. Our operations with Fourier transforms are purely formal. We are concerned with the final representation formula which will be justified for its sake. Let

$$\hat{f}_1(w) = \frac{1}{\pi}\int_0^\infty f_1(\zeta)\cos w\zeta d\zeta, \qquad \hat{f}_2(w) = \frac{1}{\pi}\int_0^\infty f_2(\zeta)\cos w\zeta d\zeta. \tag{14}$$

Note that the possibility that these integrals diverge does not concern us. Let

$$w_2(x, y) = \int_{-\infty}^\infty (A(w)\exp(iwy - wx) + B(w)\exp(iwy - w(\pi - x)))dw. \tag{15}$$

Then by (10) and (11), and formally using the Fourier inversion formula, we get

$$w_2(0, y) = 2\int_0^\infty \hat{f}_1(w)\cos wydw, \tag{16}$$

$$w_2(\pi, y) = 2\int_0^\infty \hat{f}_2(w)\cos wydw. \tag{17}$$

Equating

$$w_2(0, y) = f_1(y), \qquad w_2(\pi, y) = f_2(y),$$

gives

$$A(w) + B(w)\exp(-w\pi) = \hat{f}_1(w), \tag{18}$$

$$A(w)\exp(-w\pi) + B(w) = \hat{f}_2(w). \tag{19}$$

Hence

$$A(w) = \Delta(w)(\hat{f}_1(w) - \hat{f}_2(w)\exp(-w\pi)), \tag{20}$$

$$B(w) = \Delta(w)(\hat{f}_2(w) - \hat{f}_1(w)\exp(-w\pi)), \tag{21}$$

where

$$\Delta(w) = (1 - \exp(-2w\pi))^{-1}. \tag{22}$$

In view of (15), (20), (21), we have

$$w_2(x, y) = \frac{2}{\pi}\int_0^\infty (f_1(\zeta)D(x, y; \zeta) + f_2(\zeta)E(x, y; \zeta))d\zeta, \tag{23}$$

where

$$\begin{aligned} D(x, y) = \int_0^\infty &\Delta(w)(\exp(-wx) - \exp(-2w\pi + wx)) \\ &\cdot \cos wy \cos w\zeta d\zeta, \end{aligned} \tag{24}$$

$$\begin{aligned} E(x, y) = \int_0^\infty &\Delta(w)(\exp(-w(\pi - x)) - \exp(-w(\pi + x))) \\ &\cdot \cos wy \cos w\zeta d\zeta. \end{aligned} \tag{25}$$

LEMMA 1. (i) w_2 *is harmonic in the open strip.*

(ii) $\lim_{y\to 0} w_2(x, y)$ *exists for* $0 \leqq x \leqq \pi$ *and defines a function continuous on* $[0, \pi]$.

(iii) $\lim_{y\to 0} w_{2,y}(x, y) = 0$, $0 < x < \pi$.

(iv) $\lim_{x\to 0} w_2(x, y) = f_1(y)$, $\lim_{x\to\pi} w_2(x, y) = f_2(y)$.

PROOF. The proof of (i), (ii), (iii) is straightforward. The proof of (iv) proceeds by approximating the kernels D and E by the Poisson kernel. Details are omitted.

In view of Lemma 1 and equation (13), the boundary value problem (1)–(6) reduces to the following integral equation

$$u(x) = \frac{1}{\pi}\int_0^\pi G[z; u(z)]K(x, z)dz + w_2(x, 0), \qquad 0 \leqq x \leqq \pi, \tag{26}$$

where $u(x) = w_1(x, 0)$ and

$$K(x, z) = \log \frac{1 - \cos(x - z)}{1 - \cos(x + z)} \cdot \tag{27}$$

Define

$$Tu(x) = \frac{1}{2\pi}\int_0^\pi G[z; u(z)]K(x, z)dz + w_2(x, 0), \qquad 0 \leqq x \leqq \pi. \tag{28}$$

Then u is a solution of (26) if and only if u is a fixed point of T. In the sequel, our arguments will be couched in the language of fixed point theorems. Let E be the Banach space of continuous functions on $[0, \pi]$. Then it is clear that T takes E into E. It is also clear that T is completely continuous, since the function $\log(1 - \cos(x \pm z))$ is integrable in z.

LEMMA 2. *T has at most one fixed point.*

The proof of this lemma proceeds exactly as in the proof of Theorem 4 of [1].

LEMMA 3. *Let $T_\lambda = \lambda T$, $0 < \lambda \leqq 1$. If u is a fixed point of T_λ, then*

$$\inf_x \inf(f_1(x), f_2(x)) \leqq u \leqq c.$$

PROOF. Let

$$v(x, y) = \frac{\lambda}{\pi} \int_0^\pi \log \frac{1 - 2\exp(-y)\cos(x - z) + \exp(-y)}{1 - 2\exp(-y)\cos(x + z) + \exp(-y)} \cdot G[z; w(z, 0)]\, dz + \lambda w_2(x, y).$$

Then, by Lemma 1,

$$\lim_{x \to 0} v(x, y) = \lambda f_1(y),$$

$$\lim_{x \to \pi} v(x, y) = \lambda f_2(y),$$

$$-\lim_{y \to 0} \frac{\partial v}{\partial y}(x, y) = G[x, u(x)].$$

To prove the double inequality of the lemma, we use the maximum modulus principle for harmonic functions.

(i) *u is $\leqq c$.*

Suppose on the contrary that $u(x) > c$ for some x. Let x_M be such that $u(x_M) = \max u(x)$. Then $G[x_M; u(x_M)] < 0$ by the definition of G, since $u(x_M) > c$. On the other hand, v is not constant. Hence, by the maximum principle,

$$-y^{-1}(v(x_M, y) - v(x_M, 0)) > 0, \qquad y > 0.$$

Since $-v_y(x_M, 0) = G[x_M, u(x_M)]$, we have a contradiction. This contradiction proves (i).

(ii) u is $\geqq \inf_x \inf(f_1(x), f_2(x)) \equiv a$.

Suppose on the contrary that for some x, we have $u(x) < a$. Let x_m be such that $u(x_m) \equiv \inf u(x)$. Then $u(x_m) = \inf v(x, y)$ by the maximum principle. The proof proceeds on the same lines as in part (i).

We now state our existence theorem.

THEOREM 1. *T has a unique fixed point u. Furthermore, $a \leqq u \leqq c$.*

PROOF. Suppose that T has no fixed point. Then by a theorem of Schaefer [2], there exist a sequence (u_n) of elements of E and a sequence of real numbers $0<\lambda_n<1$, such that

$$u_n = \lambda_n T u_n, \qquad \|u_n\| \to \infty.$$

But by Lemma 3, $a \leqq u_n \leqq c$. This contradiction proves that T has a fixed point u, which is unique by Lemma 2. Furthermore, $a \leqq u \leqq c$, by Lemma 3. Thus the theorem is proved.

THEOREM 2. *The boundary value problem* (1)–(6) *has a unique solution.*

PROOF. Each fixed point of the operator T gives rise to a solution of the boundary value problem (1)–(6) in an obvious way. Hence by Theorem 1, the given boundary value problem has a solution. Let $v(x, y)$, $v'(x, y)$ be two solutions of the boundary value problem. Let $w(x, y) = v(x, y) - v'(x, y)$. Then w vanishes on the edges $x=0$, $0 \leqq y < \infty$, and $x=\pi$, $0 \leqq y < \infty$. Using the maximum modulus principle, and the properties of the function $G[x, u(x)]$, we can prove that w vanishes on the base $0 \leqq x \leqq \pi$. Then, again by the maximum modulus principle, w must be the null function. This proves the theorem.

The problem of actually constructing the solution of the problem is one of great interest. If T is a contraction, i.e., if $\|Tu - Tu'\| \leqq \alpha \|u - u'\|$, $0<\alpha<1$, then the fixed point of T can be obtained by successive approximation. We shall show that even when T is only nonexpansive, i.e., $\|Tu - Tu'\| \leqq \|u - u'\|$, the fixed point of T can still be obtained by successive approximation.

THEOREM 3. *Let T be nonexpansive. Then the fixed point u of T can be obtained by successive approximation. More precisely, let $T_n = \alpha_n T$, $0<\alpha_n<1$.*

If $\alpha_n \to 1$ and $\alpha_n^n \to 0$, then $\lim_n T_n^n w = u$ *for any w in E.*

PROOF. By T_n^n we mean of course the nth iterate of T_n. For each n, T_n is a contraction, and hence has a fixed point u_n, $u_n = T_n u_n$.

By Lemma 3, the sequence (u_n) is bounded, and, by the above relation, is relatively compact. Hence (u_n) has a cluster value. Now, each cluster value of (u_n) is a fixed point of T. Since by Theorem 1, T has a unique fixed point, it follows that (u_n) has a unique cluster value u. Since (u_n) is relatively compact, (u_n) converges to u. Now, if w is any element of E,

$$\begin{aligned} \|T_n^n w - u_n\| &= \|T_n^n w - T_n^n u_n\| \\ &\leqq \alpha_n^n \|w - u_n\|. \end{aligned}$$

Since (u_n) is bounded, the theorem follows.

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