A COUNTEREXAMPLE IN WEIGHTED MAJORITY GAMES

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1. Two ideas have been put forward for distinguishing a unique system of weights "naturally" for each weighted majority game. Let us describe them (insufficiently) as (a) taking minimal integral weights and (b) taking minimal weights such that every winning set wins by a margin of at least one unit. For the sets A, B of solutions of problems (a) and (b), two possibilities have been known. They may be the same single point; B may be the convex hull of a set A of several points.

This note exhibits a third possibility: distinct single points.

B. Peleg has introduced idea (b) and described a particular way of selecting a unique point in B [2]. It is not known whether Peleg's point (the "nucleolus") can differ from the centroid of B. Peleg asked whether, when (a) yields a single point, it is his point. We see that no refinements of ideas (a) and (b) agree.

2. There is no difficulty in finding examples for which A and B are the same single point, but let us notice 135 specific examples: all strong weighted majority games of at most seven players. They are listed, with the distinguished weights, in [1], with slight indications of how one shows that these are the unique solutions of problem (a). In fact all my arguments were by linear inequalities and thus specifically concerned with problem (b). Of course, when problem (b) has a unique solution and it is integral, it is the unique solution of (a). (That is the only general relation I know between the two solution sets.)

There is a typographical error in the list in [1]; in the first line of p. 28, the weights (1, 1, 1, 1, 2, 2, 3) should end "5," not "3."

In [1], I considered the game (1, 3, 5, 6, 8, 11, 12, 23, 28, 31, 31, 38), and sketched the proof of the following. Call the given weights $a_1, \cdots, a_{12}$. Any integral weights w_j for this game are at least as great as a_j, for $j \leqq 11$, and $w_{12} \geqq 37$. However, w_{12} can be 37, with $w_4 = 7$, so problem (a) has at least these two solutions (since $(1, \cdots, 6, \cdots, 37)$ does not give weights for the game). Again the arguments are linear and apply to problem (b). Moreover, it is easy to see that any weights v_j making each winning set win by at least a margin of 1

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must satisfy $v_4+v_{12}\geqq 44$, so that v_4+v_{12} with either $v_1+v_8+v_{11}$ or $v_2+v_3+v_5+v_6+v_9$ will exceed the rest by 1. Thus problem (a) has only the two solutions, and (b), the segment joining them.

3. For the new example, the unique minimal integral weights $r_1, \cdots, r_{19}$ are (1, 1, 1, 6, 11, 17, 35, 38, 45, 56, 101, 112, 157, 213, 355, 284, 284, 284, 284). The minimum "unit margin" weights $s_1, \cdots, s_{15}$ are (1, 1, 1, 5, 9, 14, 29, 31, 37, 46, 83, 92, 129, 175, 292); the last four are each 467/2.

The s_i are weights for a game because no subset of $\{s_1, \cdots, s_{15}\}$ adds up to 239 (as one may see). Thus the tie one would expect from $3\cdot 467/2+239=467/2+706$ does not occur.

One must check first that $(s_1, \cdots, s_{15})$ gives the unique solution of problem (b) (hence also of (a)) for the fifteen-player game it defines. This would naturally follow from a triangular array of fifteen inequalities $w_1\geqq 1, \cdots, w_{15}\geqq w_1+w_5+w_6+w_{10}+w_{12}+w_{13}+1$, for unit margin weights w_j. To get each of these, one exhibits a third disjoint set S of players which, with the jth player, wins, but with the set indicated on the right of the jth inequality, loses. Twelve of these ($j\neq 4$, 6, 14) are handily found. We want to refer to these inequalities I_j again; we list them, putting the numbers s_i for the symbols "w_i" and omitting the units. Three times $1>0$; $9>5+1+1+1$, $29>14+9+5$, $31>29+1$, $37>31+5$, $46>31+14$, $83>37+31+14$, $92>46+31+14$, $129>92+31+5$, $292>129+92+46+14+9+1$. For the other three one easily checks that $29+5>31+1+1$, $14+1>9+5$, and $175+1>129+46$ serve the purpose.

It follows at once that $(s_1, \cdots, s_{19})$ is the unique solution of problem (b) for the game it defines.

Basically, the s_i and the r_i define the same game because the point $(r_1-s_1, \cdots, r_{15}-s_{15})$ is a limit point of the open cone of all systems of weights for that game, i.e. every winning set has at least half of the $(r-s)$-weight. (As one may see, it is necessary also to check what happens to sets of s-weight>239 or <239.)

One can show that the r's give the unique solution to problem (a) as follows. First, I need a special check that for integral weights v_i, if v_1 or v_2 or v_3 exceeds 1 then all $v_i\geqq r_i$. The sum of any two of v_1, v_2, v_3 must exceed the third; so I_4 and I_8 yield $v_4\geqq 7$, and so on. Next, having $v_1=v_2=v_3=1$, one can replace I_6, I_7, and $I_9, \cdots, I_{15}$ by equalities ($9+5+1>14$ etc., $14+9+5+1+1>29$ etc.). Next, use the relation $46+1+1+1>29+14+5$ with sharp I_{10} and $29+5>31+1+1$ to evaluate v_8 as v_7+v_4-3. Then $v_1, \cdots, v_{15}$ are determined by v_4 and v_5. To show that $v_4\geqq 6$, note that we already have

$v_4 \geqq 5$ and $v_5 \geqq 9$. Putting $v_4 = 5$ and $v_5 = 9 + t$, we can compute the rest and find $v_1 + \cdots + v_{15} = 945 + 60t$. Since the players 16, 17, 18 and 19 win with 1 and 4 but lose with 4 alone, and $v_{16} = v_{17} = v_{18} = v_{19}$ (since all two-to-two divisions of them are alike, and players of weight 1 matter), and $60 \equiv 0 \pmod 4$, v_{16} is not an integer. As this sole nonlinear bit of argument is crucial, let us spell out: $4v_{16} + v_4 = v_2 + v_3 + v_5 + \cdots + v_{15}$, since either side would exceed with v_1 added. So $4v_{16} = 934 + 60t$, which is absurd.

Having $v_4 \geqq 6$, and inferring $v_5 \geqq 11$ from $v_7 + v_5 > v_8 + v_4 + 1$ $(29 + 9 > 31 + 5 + 1)$, the rest is immediate.

References

1. J. Isbell, *On the enumeration of majority games*, MTAC **13** (1959), 21–28.

2. B. Peleg, *On weights of constant-sum majority games*, SIAM J. Appl. Math. **16** (1968), 527–532.

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