THE SET OF IRREDUCIBLE OPERATORS IS DENSE

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Let $\mathcal{H}$ be a separable (finite or infinite-dimensional) complex Hilbert space. P. R. Halmos [1] has shown that the set of irreducible operators on $\mathcal{H}$, (i.e., operators with no nontrivial reducing subspaces), is uniformly dense in the space of bounded operators on $\mathcal{H}$. In this note we give a very simple proof of Halmos's theorem.

Let A be any bounded operator on $\mathcal{H}$ and let $\epsilon > 0$. By the spectral theorem there exists a Hermitian operator D whose matrix is diagonal with respect to an o.n. basis $\{e_n\}$ such that

$$\|D - (A + A^*)/2\| < \frac{\epsilon}{4} \cdot$$

Then there is a Hermitian operator D_1 diagonal with respect to $\{e_n\}$ such that all the eigenvalues of D_1 are distinct and $\|D - D_1\| < \epsilon/4$. Now let D_2 be any Hermitian operator within $\epsilon/2$ of $(A - A^*)/2i$ whose matrix with respect to $\{e_n\}$ has all entries different from 0; (such operators obviously exist in profusion). Then the operator $D_1 + iD_2$ is within ϵ of A. Also $D_1 + iD_2$ is irreducible, since the invariant subspaces of D_1 are the subspaces spanned by subcollections of $\{e_n\}$, and none of these are invariant under D_2 except $\{0\}$ and $\mathcal{H}$.

REFERENCE

1. P. R. Halmos, *Irreducible operators*, Michigan Math. J. 15 (1968), 215–223.

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