A NOTE ON THE HOLONOMY GROUP OF MANIFOLDS WITH CERTAIN STRUCTURES

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1. **Introduction.** There has recently been considerable interest in manifolds carrying a tensor field h of type (1, 1) which satisfies certain conditions. The reader should see any of the references [1], [3], [4], or [5] listed at the end of this paper. A typical restriction is that h satisfies some algebraic condition, such as the assumption that h has distinct eigenvalues. Integrability conditions on h constitute yet another class of restrictions that might be imposed. For example, one might require either that the Nijenhuis tensor $[h, h]$ be zero or the stronger condition that h has a vanishing covariant derivative with respect to any vector field.

In this paper some properties of the holonomy group of a manifold with certain structures are proved. One such result is contained in Theorem 3.1 where it is assumed that there is a cyclic h with the property that $\nabla_X h = 0$ for every vector field X. It is then shown that if e_0 is a generator for h, then elements of the holonomy group give rise to other generators for h. If the additional restriction that h has a vanishing Lie derivative is imposed, then Theorem 3.4 yields the result that the manifold is flat.

2. **Notation and definitions.** Let M be a Riemannian manifold of dimension $(n+1)$. Covariant differentiation with respect to the Riemannian connection on M will as usual be denoted by ∇, and $\mathfrak{L}$ will denote Lie differentiation. The manifold M is said to be *flat* if and only if the curvature R vanishes; that is, if and only if

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z = 0$$

for any three vector fields X, Y and Z on M.

The holonomy group on M is defined as follows. Let $m \in M$ and $p = (m, e_0, \cdots, e_n) \in F(M)$, the principal $O(n+1)$-bundle of orthonormal frames over M where $O(n+1)$ is the orthogonal group. Let γ: $[0, 1] \to M$ be a closed piecewise differentiable curve starting and ending at m. Let τ_γ denote parallel translation around γ with respect to the Riemannian connection. The *holonomy group* at m of the Riemannian connection is denoted by Φ_m and defined by

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$$\Phi_m = \{g \in O(n+1) \mid pg = \tau_\gamma(p) \text{ for some } \gamma\}.$$

The statement that $(m, e_0', \cdots, e_n') = (m, e_0, \cdots, e_n)g$ will be denoted by $e_i' = ge_i$, and γ^* is our notation for the tangent vector field to γ.

Let M_m denote the tangent space at m. A tensor field h of type (1, 1) is said to be *cyclic* if and only if the minimal and characteristic polynomials of h are identical. Thus if it is assumed that h is nonsingular at a point then there exists a generator $e_0 \in M_m$ such that $\{h^i e_0: 0 \leq i \leq n\}$ is a basis of M_m.

Finally, it should be remarked that if h is singular at a point, then at this point a nonsingular tensor field h^* can be found such that $\nabla_X h = \nabla_X h^*$ for every vector field X. The tensor field h^* is obtained by setting $h^* = h + \alpha I$, where α is a suitable nonzero constant.

3. Some properties of the holonomy group.

THEOREM 3.1. *If there exists on M a cyclic tensor field h of type* (1, 1) *with generator e_0 at $m \in M$ such that*

$$\{e_i = h^i e_0 \mid i = 0, 1, \cdots, n\}$$

is a basis of the tangent space M_m at m, and if $\nabla_X h = 0$ for every vector field X, then ge_0 is also a generator for h for every $g \in \Phi_m$.

PROOF. Let $Y_i(t)$, $i = 0, 1, \cdots, n$ be parallel vector fields along γ with $Y_i(0) = e_i$ and $Y_i(1) = e_i'$; that is, $\nabla_{\gamma^*} Y_i = 0$ and $Y_i(0) = e_i$, and hence $Y_i(1) = e_i'$. It should first be observed that hY_0 is parallel along γ since

$$\nabla_{\gamma^*} hY_0 = (\nabla_{\gamma^*} h) Y_0 + h(\nabla_{\gamma^*} Y_0) = 0.$$

Now $hY_0(0) = he_0 = e_1$ and hence by the uniqueness of solutions of the differential equations $\nabla_{\gamma^*} hY_0 = 0$, $hY_0(1) = Y_1(1) = e_1'$. One can then proceed inductively. If $h^{i-1}Y_0$ is parallel, then $\nabla_{\gamma^*} h^i Y_0 = (\nabla_{\gamma^*} h) h^{i-1} Y_0 + h(\nabla_{\gamma^*} h^{i-1} Y_0) = 0$. Now $h^i Y_0(0) = e_i$ and therefore by the uniqueness of solutions $h^i Y_0(1) = Y_i(1)$. Hence $Y_0(1) = ge_0$ is a generator for every $g \in \Phi_m$.

LEMMA 3.2. *If there exists on M a cyclic tensor field h of type* (1, 1) *such that $\nabla_X h = 0$ for every vector field X, then the coefficients of the characteristic polynomial of h are constant.*

PROOF. Note first that $\nabla_X h^i = 0$ for any nonnegative integer i and any vector field X. If the vector field X_0 is a generator for h and if h has a characteristic polynomial

$$h^{n+1} = a_0 I + a_1 h + \cdots + a_n h^n,$$

then $\nabla_X h^{n+1}X_0 - h^{n+1}\nabla_X X_0 = (\nabla_X h^{n+1})X_0 = 0$. Denote $h^i X_0$ by X_i and observe that $\{X_i: 0 \leq i \leq n\}$ is a basis of vector fields. A short calculation will then establish the result that

$$\nabla_X h^{n+1}X_0 = \sum_{i=0}^{n} (Xa_i)X_i + h^{n+1}\nabla_X X_0,$$

and consequently that $\sum_{i=0}^{n} (Xa_i)X_i = 0$. Hence $X(a_i) = 0$ implies $a_0, a_1, \cdots, a_n$ are all constant.

COROLLARY. *If there exists on M a tensor field h of type $(1, 1)$ with distinct eigenvalues $\lambda_0, \cdots, \lambda_n$ such that $\nabla_X h = 0$ for every vector field X, then the eigenvalues $\lambda_0, \cdots, \lambda_n$ are all constant.*

The corollary follows immediately from the facts that any h with distinct eigenvalues is necessarily cyclic and that the coefficients a_i of the characteristic polynomial of h are symmetric functions of the eigenvalues.

THEOREM 3.3. *If there exists on M a tensor field h of type $(1, 1)$ with distinct eigenvalues $\lambda_0, \cdots, \lambda_n$ with corresponding eigenvectors $\{e_i\}$ at $m \in M$, and if $\nabla_X h = 0$ for every vector field X, then $\{ge_i\}$ is also a set of eigenvectors of h for every $g \in \Phi_m$, and moreover g is a diagonal matrix with elements $+1$ or -1 on the main diagonal.*

PROOF. Let $Y_i(t)$ be vector fields as in the proof of Theorem 3.1 and compute $\nabla_{\gamma^*} h Y_i$. One obtains the result that

$$\nabla_{\gamma^*} h Y_i = (\nabla_{\gamma^*} h) Y_i + h(\nabla_{\gamma^*} Y_i) = 0$$

and hence hY_i is parallel along γ. Now $hY_i(0) = he_i = \lambda_i e_i = \lambda_i Y_i(0)$, but the eigenvalues λ_i are constant so that $he_i' = \lambda_i e_i'$ where $e_i' = Y_i(1)$, and $Y_i(1)$, by our choice, is ge_i. Finally, since the λ_i are also eigenvalues for e_i, g must be a diagonal matrix with elements $+1$ or -1 on the main diagonal.

If h satisfies the additional condition that its Lie derivative vanishes, then M is flat. This fact is proved in the following theorem.

THEOREM 3.4. *If there exists on M a cyclic tensor field h of type $(1, 1)$ such that $\nabla_X h = \mathfrak{L}_X h = 0$ for every vector field X, then M is flat.*

PROOF. It should first be observed that

$$(\mathfrak{L}_X h) Y = 0 = (\nabla_X h) Y + h(\nabla_{hY} X) - \nabla_{hY} X$$

for any vector fields X and Y, and hence $\nabla_{hY} X = h\nabla_Y X$. If X_0 is a generator for h, let $X_i = h^i X_0$ so that the set $\{X_0(m), \cdots, X_n(m)\}$

is a basis of M_m. It is clear that $[X_i, X_j] = -h^{i+j}\mathfrak{L}_{X_0}X_0 = 0$, and hence

$$\begin{aligned} R(X_i, X_j)X_k &= \nabla_{X_i}\nabla_{X_j}X_k - \nabla_{X_j}\nabla_{X_i}X_k \\ &= h^i\nabla_{X_0}(h^{j+k}\nabla_{X_0}X_0) - h^j\nabla_{X_0}(h^{i+k}\nabla_{X_0}X_0) \\ &= 0, \end{aligned}$$

which completes the proof of the theorem.

It should be remarked that Theorem 3.4 remains valid if the hypothesis that h is cyclic is replaced by the hypothesis that h has distinct eigenvalues.

References

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