3-MANIFOLDS FIBERING OVER S^1 WITH NONUNIQUE CONNECTED FIBER

JEFFREY L. TOLLEFSON[1]

The purpose of this note is to show that even in a class of very simple Seifert spaces fibered over the circle in the sense of Stallings [3], the fiber is not unique. The examples presented are 3-manifolds of the form $T(g)\times S^1$ where $T(g)$ denotes a closed connected orientable 2-manifold of genus g, $g\geq 2$. In particular we show that each of these manifolds fiber over the circle in an infinite number of essentially distinct ways.

THEOREM. *Let $g\geq 2$. For every integer $n\geq 0$ there is a fibering of $T(g)\times S^1$ over the circle with fiber $T(m)$ where $m=g+n(g-1)$.*

PROOF. Assume $T(m)$ is in 3-space with $n+1$ handles of genus $g-1$ symmetric about one hole as pictured below for the case when $g=3$ and $n=2$, i.e. $T(7)$.

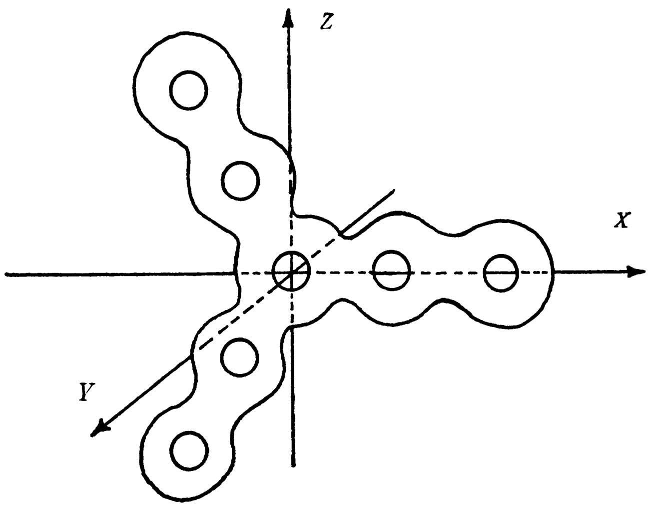

FIGURE 1

Let $h: T(m)\rightarrow T(m)$ be the homeomorphism given by a rotation of $2\pi/n+1$ degrees about the y-axis. h generates a free cyclic group action on $T(m)$ of order $n+1$. Let M be the 3-manifold obtained from $T(m)\times I$ by pasting the boundary components $T(m)\times\{0\}$ and $T(m)\times\{1\}$ together by the homeomorphism h, or equivalently, let $M=T(m)\times R/\{(x, t)\sim(h(x), t+1)\}$ where R denotes the real num-

Received by the editors June 3, 1968.

[1] The author was supported by a NASA Traineeship.

bers. Write points of M as $[x, t]$. We wish to define a free SO(2) action on M. Consider SO(2) as R/Z with elements written as $[s]$ for $s \in R$. Define an action of SO(2) on M by

$$[s] \times [x, t] \to [x, t + (n+1)s].$$

It is easy to check that this is a well-defined free action.

By [1, Theorem 2] M is homeomorphic to a manifold $\overline{M}$ with a standard SO(2) action $\{b; (0, g, 0, 0)\}$ where g denotes the genus of the orbit space $\overline{M}/\mathrm{SO}(2)$. Briefly $\overline{M}$ can be described as follows. Let SO(2) operate on $T(g) \times S^1$ by

$$e^{i\phi} \times (m \times e^{i\psi}) \to (m \times e^{i\phi} e^{i\psi}).$$

Remove an invariant tubular neighborhood generated by the interior of a closed disk neighborhood D in $T(g)$. Now equivariantly sew the solid torus $D \times S^1$ back in by a homeomorphism matching principal orbits but sending a cross-section meridian m on $D \times S^1$ to a cross-sectional curve q on $\mathrm{Bd}(T(g) \times S^1 - D \times S^1)$ satisfying the homology relation $q \sim m' + bh$ where h is a principal orbit and m' a meridianal curve. The resulting space is $\overline{M}$ with the standard action.

Using Van Kampen's theorem we find that $H_1(\overline{M}, Z) = \oplus_{2g} Z \oplus Z_b$ where $Z_b = (x : x^b = 1)$. Moreover Z_b is generated by a principal orbit h. But $\overline{M}$ fibers over the circle if and only if the order of the element in $H_1(\overline{M}; Z)$ represented by a principal orbit is infinite [2, Satz 8]. By the original construction of M it is clear that M, and hence $\overline{M}$, fibers over the circle. Therefore we must have $b = 0$. This proves the theorem since $\{0; (0, g, 0, 0)\}$ is precisely $T(g) \times S^1$.

An interesting result on groups follows immediately. From the fibering over S^1 of $T(g) \times S^1$ with fiber $T(m)$, $m = g + n(g-1)$, we get the exact sequence

$$0 \to \pi_1(T(m)) \to \pi_1(T(g) \times S^1) \to \pi_1(S^1) \to 0.$$

Let $G(g) = \pi_1(T(g)) = (x_1, \cdots, x_{2g} : [x_1, x_2][x_3, x_4] \cdots [x_{2g-1}, x_{2g}] = 1)$. Then for every $n \geqq 0$ we have $G(m)$ contained in $G(g) \times Z$ as a normal subgroup such that $G(g) \times Z/G(m) = Z$.

References

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Michigan State University and
Tulane University