EXTENDED MALCEV DOMAINS

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In 1936, Malcev [1] constructed a cancellative monoid M generated by eight elements which couldn't be embedded in a group. Later on, Chehata [2] and Vinogradov [3] showed that M could be ordered. This gave a counterexample to the conjecture that every ordered monoid could be embedded in a group. Since the monoid ring $F[M]$ of an ordered monoid M over a (not-necessarily commutative) field F is a (not-necessarily commutative) integral domain, this also showed that not every integral domain could be embedded in a field. In the present paper, we give a somewhat different construction of an ordered monoid N generated by six or more elements which cannot be embedded in a group.

Let Z denote the ring of integers, $S=\{x_{ij} \mid i=1, 2, j\in Z\}$ be a set of indeterminates, and A be the free monoid generated by S. Each $a\in S$, $a\neq 1$, has the form $a=a_1 \cdots a_n$ for some $a_i\in S$. We call n the *degree* of a, $n=\deg a$. As usual, we let $\deg 1=0$. We order A by degrees and lexicographically from the assumed ordering

$$1 < x_{1i} < x_{1j} < x_{2i} < x_{2j} \qquad \text{for all } i, j \in Z \text{ with } i < j.$$

Thus, if $a=a_1 \cdots a_m$ and $b=b_1 \cdots b_n$ with a_i, $b_i\in S$, then $a<b$ iff either $\deg a<\deg b$ or $\deg a=\deg b$ and there exists an integer k such that $a_i=b_i$ if $i<k$ and $a_k<b_k$.

For any $i, j\in Z$, let $[i, j]$ and $[i, j]'$ be defined as follows: (1) $[i, j]=[i, j]'=(i+j)/2$ if $i+j$ is even, (2) $[i, j]=(i+j-1)/2$ and $[i, j]'=(i+j+1)/2$ if $i+j$ is odd. Also, let $T\subset A$ be defined by

$$T = \{x_{2i}x_{1j} \mid i, j \in Z, i > j\}$$

and B be the ideal of A generated by T. Finally, let $N=A-B$.

We define an operation of multiplication, $\cdot$, in N by letting $a\cdot b=ab$ if $ab\notin B$, and

$$x_{2i}\cdot x_{1j} = x_{2k}x_{1k'} \qquad \text{if } i > j, \quad \text{where} \quad k = [i, j].$$

Thus, if $a=a'x_{2i}$ and $b=x_{1j}b'$, with $i>j$ and a', $b'\in N$, we have $a\cdot b=a'x_{2k}x_{1k'}b'$. Since nothing more happens in $a\cdot b$ than the replacement of one x_{2i} on the right end of a by x_{2k} and one x_{1j} on the left end of b by $x_{1k'}$, evidently multiplication in N is associative and $\deg a\cdot b=\deg a+\deg b$ for all $a, b\in N$.

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Each $a \in N$, $a \neq 1$, has a unique normal form $a = a_1 \cdots a_n$ where all $a_i \in S$ and $a_i a_{i+1} \notin T$, $i = 1, \cdots, n-1$. We order N by assuming its elements, in normal form, are ordered by the ordering in A.

THEOREM 1. $\{N; \cdot\}$ *is an ordered monoid.*

PROOF. Let $a, b, c \in N$ with $a < b$. If $\deg a < \deg b$, then clearly $a \cdot c < b \cdot c$ and $c \cdot a < c \cdot b$. So let us assume that $\deg a = \deg b = n$ and $c \neq 1$.

Case 1. $n > 1$. By assumption, $a = a_1 \cdots a_n$, $b = b_1 \cdots b_n$, $c = c_1 \cdots c_k$, where $a_i, b_i, c_i \in S$, and there exists an integer j such that $a_i = b_i$ if $i < j$ and $a_j < b_j$. If $j = 1$, evidently $a \cdot c < b \cdot c$, and we need only check that $c_k \cdot a_1 < c_k \cdot b_1$ to prove that $c \cdot a < c \cdot b$. If $1 < j < n$, $c_k \cdot a_1 = c_k \cdot b_1$ and hence $c \cdot a < c \cdot b$; and obviously $a \cdot c < b \cdot c$. If $j = n$, $c_k \cdot a_1 = c_k \cdot b_1$ and $c \cdot a < c \cdot b$; while $a \cdot c < b \cdot c$ provided $a_n \cdot c_1 < b_n \cdot c_1$. This reduces the problem to the following case.

Case 2. $n = 1$, with $a, b, c \in S$. If $a = x_{1i}$ and $b = x_{1j}$, with $i < j$, then $a \cdot c = ac < bc = b \cdot c$. Also, $c \cdot a = ca < cb = c \cdot b$ unless $c = x_{2m}$ and $m > i$. If $i < m \leqq j$, then $c \cdot a = x_{2k} x_{1k'} < x_{2m} x_{1j} = c \cdot b$ since $k = [m, i] < m$. If $m > j$, $c \cdot a = x_{2k} x_{1k'} < x_{2l} x_{1l'}$ because either $k = [m, i] < [m, j] = l$ or $k = l$ and $k' < l'$ (this is the case only if $j = i+1$ and $m+i$ is even).

If $a = x_{2i}$ and $b = x_{2j}$, with $i < j$, then a similar analysis shows that $c \cdot a < c \cdot b$ and $a \cdot c < b \cdot c$ for all $c \in S$.

Finally, we might have $a = x_{1i}$ and $b = x_{2j}$. Then $a \cdot c < b \cdot c$ for all $c \in S$. If $c = x_{1m}$ then clearly $c \cdot a = ca < cb = c \cdot b$. If $c = x_{2m}$, then $c \cdot a = ca < cb = c \cdot b$ if $m \leqq i$, whereas, if $m > i$, $c \cdot a = x_{2k} x_{1k'} < x_{2m} x_{2j} = c \cdot b$ because $k = [m, i] < m$. This proves the theorem.

For any $m, n \in Z$, with $m < n$, let $S(m, n) = \{x_{ij} \mid i = 1, 2, m \leqq j \leqq n\}$ and $N(m, n)$ be the submonoid of N generated by $S(m, n)$. Since $i \leqq [i, j] \leqq [i, j]' \leqq j$ for all $i, j \in Z$ with $i < j$, evidently each element of $N(m, n)$ when expressed in normal form is a product of elements of $S(m, n)$. If $m < n$, $m' < n'$, and $n - m = n' - m'$, then it is clear that $N(m, n) \cong N(m', n')$.

THEOREM 2. *The monoid* $N(1, 3)$ *generated by six elements cannot be embedded in a group.*

The standard proof of Malcev can be used. Thus, assume that $N(1, 3)$ is a submonoid of some group G. Since $x_{22} \cdot x_{11} = x_{21} x_{12}$, $x_{23} \cdot x_{11} = x_{22} x_{12}$, and $x_{23} \cdot x_{12} = x_{22} x_{13}$, we have in G that $x_{22}^{-1} x_{21} = x_{11} x_{12}^{-1} = x_{23}^{-1} x_{22} = x_{12} x_{13}^{-1}$. Hence, $x_{22} x_{12} = x_{21} x_{13}$ contrary to the definition of the monoid N. Therefore, $N(1, 3)$ cannot be embedded in a group.

A consequence of Theorem 2 and our remarks above is that the monoid $N(m, n)$ with $n > m+1$ cannot be embedded in a group.

If F is any integral domain, then the monoid ring $F[N]$ generated by N over F is also an integral domain. For if $f=f_1a_1+\cdots+f_ma_m$ and $g=g_1b_1+\cdots+g_nb_n$ are elements of $F[N]$, with $f_i, g_i \in F$, $f_m \neq 0$, $g_n \neq 0$, and $a_i, b_i \in N$ such that $a_1 < \cdots < a_m$ and $b_1 < \cdots < b_n$, then fg is nonzero with highest term $f_m g_n a_m b_n$. Since N cannot be embedded in a group, $F[N]$ cannot be embedded in a field. This is also true of each subdomain $F[N(m, n)]$ for which $n > m+1$.

For any integral domain D and its associated ring $(D)_n$ of $n \times n$ matrices over D, the poset P of annihilating right ideals of $(D)_n$ has dimension at least n. If D is a field, or a subring of a field, the dimension of P is exactly n. For other domains, the dimension of P might be considerably different as the following result shows.

THEOREM 3. *If F is a field, $R=F[N]$, and P is the poset of annihilating right ideals of $(R)_2$, then P satisfies neither the dcc nor the acc.*

PROOF. Let $\{e_{ij} \mid i, j=1, 2\}$ be the usual unit matrices in $(R)_2$. If $i+1 \geqq j$ and $r+1 \geqq s$, then $x_{2i} \cdot x_{1j} = x_{2r} \cdot x_{1s}$ iff $i+j=r+s$. For each $n \in Z$, let $a_n = x_{2n}e_{11} - x_{2n+1}e_{12}$ and $b_{nj} = x_{1n+1}e_{1j} + x_{1n}e_{2j}$, $j=1, 2$. Then the right annihilator of a_n in $(R)_2$ is given by

$$(a_n)^r = \sum_{k=-\infty}^{n} \sum_{j=1}^{2} b_{kj}(R)_2. \tag{1}$$

For if $\sum f_{ij}e_{ij} \in (a_n)^r$, then $x_{2n}f_{11} = x_{2n+1}f_{21}$ and $x_{2n}f_{12} = x_{2n+1}f_{22}$. If $f_{ij} = p_{ij1}c_{ij1} + p_{ij2}c_{ij2} + \cdots$, with $p_{ijk} \in F$, $c_{ijk} \in N$, and $c_{ij1} < c_{ij2} < \cdots$, necessarily $p_{11k} = p_{21k}$, $p_{12k} = p_{22k}$, $x_{2n} \cdot c_{11k} = x_{2n+1} \cdot c_{21k}$, $x_{2n} \cdot c_{12k} = x_{2n+1} \cdot c_{22k}$ for each k. Thus, it is clear that $c_{11k} = x_{1i} \cdot c'_{11k}$, $c_{21k} = x_{1j} \cdot c'_{21k}$ with $i \leqq n+1$, $j=i-1$, and $c'_{11k} = c'_{21k}$ for each k; and similarly for c_{12k} and c_{22k}. Thus, $\sum f_{ij}e_{ij}$ has the form given in (1). Since $(a_n)^r < (a_{n+1})^r$ for all $n \in Z$, P satisfies neither the *dcc* nor the *acc*. This proves the theorem.

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