ON THE ZEROS OF THE BERGMAN FUNCTION IN DOUBLY-CONNECTED DOMAINS

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The purpose of this note is to show that every doubly-connected Lu Qi-Keng domain in C^1 is pseudoconformally equivalent to a disc with the center deleted. This extends a result of M. Skwarczynski [4], who gave an example of a domain C^1 which is not a Lu Qi-Keng domain. (See definition below.) Our results indicate that, at least in this particular case, there exists a connection between the degree of connectivity of D and zeros of the Bergman function. We use the notation $z = (z^1, z^2, \cdots, z^n)$ for a point in $D \subset C^n$ and $\bar{t}$ for $(\bar{t}^1, \bar{t}^2, \cdots, \bar{t}^n)$. We denote

$$D^* = \{\bar{t} \mid t \in D\}.$$

Definitions and theorems relating to the Bergman function can be found in [1]. In what follows, the Bergman function of the domain $D \subset C^n$ will be denoted by $K_D(z, \bar{t})$.

DEFINITION. A domain $D \subset C^n$ is a Lu Qi-Keng domain if the equation $K_D(z, \bar{t}) = 0$ has no solution in $D \times D^*$ (see [4]).

THEOREM 1. *Let D be the ring $0 < r < |z| < 1$. Then D is not a Lu Qi-Keng domain.*

PROOF. As was shown by Zarankiewicz in [5], see also [1, p. 10],

$$K_D(z, \bar{t}) = \frac{1}{\pi z \bar{t}} \left[\wp\{\log(z\bar{t}); \omega, \omega'\} + \frac{\eta}{\omega} - \frac{1}{2\omega'} \right], \tag{1}$$

$\wp$ is the Weierstrassian $\wp$-function, $\omega = \pi i$, $\omega' = \log r$, 2η is the increment of the Weierstrassian ζ-function related to the half-period ω. (We note that since the first half-period $\omega = \pi i$, the value of the $\wp$-function does not depend on the value chosen for $\log(z\bar{t})$.) Using the Legendre equation $\eta\omega' - \eta'\omega = \pi i/2$ $(\text{real}(\omega'/i\omega) > 0)$, (1) can be written as

$$K_D(z, \bar{t}) = \frac{1}{\pi z \bar{t}} \left[\wp(u; \omega, \omega') + \frac{\eta'}{\omega'} \right], \tag{2}$$

$u = \log(z\bar{t})$. The function e^u maps the period-parallelogram (points

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Re $u=0$, $\log r^2$ excluded) onto the q-ring $0<r^2<|q|<1$, $q=z\bar{t}$. Since the doubly periodic function $\wp(u)$ attains every value in the period-parallelogram exactly twice, the function

$$\eta'/\omega' + \wp(u) \tag{3}$$

attains every value in the q-ring except for values attained when u is in the segment $\operatorname{Re} u=0$, $0\leqq \operatorname{Im} u\leqq \pi$. Consider the boundary of the rectangle with vertices 0, πi, $\log r+\pi i$, $\log r$, in the u-plane with counterclockwise orientation. On this boundary, $\wp$ attains real values increasing monotonically from $-\infty$ to $+\infty$. The function (3) has the same property, and we conclude that the exceptional values of (3) form a closed segment $[-\infty, \eta'/\omega'+\wp(\pi i)]$ on the real axis. We infer the Bergman function has a zero in $D\times D^*$ if and only if

$$\eta'/\omega' + \wp(\pi i) < 0. \tag{4}$$

We prove next that, for every $0<r<1$, (4) holds. Consider the new pair of primitive half-periods, $\tilde{\omega}=-\log r$, $\tilde{\omega}'=\pi i$. We then obtain

$$\eta'/\omega' + \wp(\pi i) = \tilde{\eta}/\tilde{\omega} + \wp(\pi i). \tag{5}$$

It is known that [2, p. 336],

$$\begin{aligned} \frac{\tilde{\eta}}{\tilde{\omega}} + \wp(u) = -\frac{\pi^2}{\tilde{\omega}^2}\Bigg\{\frac{1}{(z-z^{-1})^2} + \sum_{m=1}^{m=\infty} \frac{h^{2m}z^{-2}}{(1-h^{2m}z^{-2})^2} \\ + \sum_{m=1}^{m=\infty} \frac{h^{2m}z^{2}}{(1-h^{2m}z^{2})^2}\Bigg\}, \\ v = u/2\tilde{\omega}, \qquad z = e^{v\pi i}, \qquad t = \tilde{\omega}'/\tilde{\omega}, \qquad h = e^{t\pi i}\ (\operatorname{Im} t > 0). \end{aligned} \tag{6}$$

Since the right-hand side of (6) for $\tilde{\omega}=-\log r$, $\tilde{\omega}'=\pi i$, and $u=\pi i$ is negative for all $0<r<1$, (4) holds. This completes the proof of Theorem 1.

THEOREM 2. *Every doubly-connected Lu Qi-Keng domain in C^1 is pseudoconformally equivalent to a disc with the center deleted.*

PROOF. Let D be a doubly-connected Lu Qi-Keng domain in C^1. It must be pseudoconformally equivalent to one of the three following domains:

(1) a plane with the center deleted,
(2) a ring $(0<r<|z|<1)$,
(3) a disc with the center deleted.

However, the Bergman function for the domain $\{z|z\neq 0\}$ is identically zero, and the Bergman function for the ring posseses zeros by

Theorem 1. Since the class of Lu Qi-Keng domains is invariant under pseudoconformal transformations, (1) and (2) do not occur.

THEOREM 3. *For every $k \geqq 3$, there exists a domain $D \subset C^1$ of connectivity k which is not a Lu Qi-Keng domain.*

PROOF. Consider a ring R and let z_0, $\bar{t}_0$ be such that $K_R(z_0, \bar{t}_0) = 0$. We choose $k-2$ distinct points $z_1, \cdots, z_{k-2}$ in R different from z_0 and $\bar{t}_0$. Consider a domain $R_m = R - \bigcup_{j=1}^{k-2} [z \mid z \in R$ and $|z - z_j| \leqq 1/m$, m a positive integer]. By the Ramadanov Theorem [**3**], the sequence $K_{R_m}(z, \bar{t}_0)$ converges locally uniformly to $K_D(z, \bar{t}_0)$ where $D = \bigcup_{m=1}^{\infty} R_m$. Since $K_D(z, \bar{t}_0) \equiv K_R(z, \bar{t}_0)$, we conclude that for sufficiently large m, the degree of connectivity of R_m is k, and by Hurwitz's theorem the function $K_{R_m}(z, \bar{t}_0)$ has a zero in R_m.

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