A NOTE ON AHLFORS' THEORY OF COVERING SURFACES[1]

JOSEPH MILES

In his results on the theory of covering surfaces, Ahlfors [1] obtained theorems which are "unintegrated" analogues of Nevanlinna's first and second fundamental theorems in the theory of meromorphic functions. For a given function f meromorphic in the plane, Ahlfors' theory unfortunately gives relatively little information about the behavior of the restriction of f to $|z| \leqq r$ for certain exceptional values of r. While observing that his second fundamental theorem did imply Picard's theorem, Ahlfors remarked that the existence of these exceptional r-values seemed to make it impossible to deduce Nevanlinna's second fundamental theorem from his by integration. Nevanlinna [3] in his treatise on meromorphic functions also stated that the exceptional r-values of Ahlfors' theory prevented one from obtaining the integrated form of the second fundamental theorem from the unintegrated form.

Two attempts to derive the classical result from Ahlfors' second fundamental theorem have met with partial success. Both however lead to versions of the classical theorem for which there are exceptional r-values even for functions of finite order. It is the purpose of this note to show that Ahlfors' theory implies a form of the classical second fundamental theorem having no exceptional r-values for functions of finite order. The proof is in fact extremely elementary, yet seems to have been overlooked.

We remind the reader of one form of Ahlfors' second fundamental theorem.

THEOREM (AHLFORS). *Let $f(z)$ be meromorphic in $|z| < \infty$. If $a_1, a_2, \cdots, a_q$ are $q \geqq 3$ distinct elements of the Riemann sphere, then there exists $h > 0$ depending on $a_1, a_2, \cdots, a_q$ such that*

$$\sum_{\nu=1}^{q} \{S(t) - \bar{n}(t, a_\nu)\} \leqq 2S(t) + hL(t), \tag{1}$$

where $\pi S(t)$ is the area on the Riemann sphere of radius $1/2$ of the image under f of $|z| \leqq t$ with due regard being paid to multiplicity, $L(t)$ is the length on the sphere of the image of $|z| = t$, and $\bar{n}(t, a_\nu)$ is the number of distinct roots of the equation $f(z) = a_\nu$ in $|z| \leqq t$.

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(The exceptional r-values of Ahlfors' theory are those on which $L(r) \geqq S(r)^{1/2+\epsilon}$; this set can be shown to have finite logarithmic measure.)

Dinghas [2] divided (1) by t and integrated from 0 to r to obtain

$$\sum_{\nu=1}^{q} \{T_0(r) - \overline{N}(r, a_\nu)\} \leqq 2T_0(r) + h\int_0^r \frac{L(t)}{t}\, dt,$$

where $T_0(r) = \int_0^r (S(t)/t)dt$ is the spherical characteristic of f and $\overline{N}(r, a_\nu) = \int_0^r (\bar{n}(t, a_\nu)/t)dt$. (There are trivial modifications if $f(0) = a_\nu$ for some ν.) He then showed

$$\int_0^r \frac{L(t)}{t}\, dt \leqq (T_0(r))^{1/2} \log T_0(r)$$

for all $r \notin E$ where $\int_E dr/r \log r < \infty$. Thus it was shown that Ahlfors' second fundamental theorem implies a weakened form of Nevanlinna's theorem in which there are exceptional r-values for functions of both finite and infinite order.

Wille [4] has shown that Ahlfors' inequality (1) implies that given $\epsilon > 0$, there exists a set E_1 of finite logarithmic measure such that

$$\sum_{\nu=1}^{q} \{T_0(r) - \overline{N}(r, a_\nu)\} \leqq (2 + \epsilon)T_0(r)$$

for all $r \notin E_1$. We now prove the following theorem.

THEOREM. *Let $f(z)$ be a transcendental meromorphic function in $|z| < \infty$. Let $a_1, a_2, \cdots, a_q$ be distinct elements of the Riemann sphere. Then Ahlfors' inequality* (1) *implies*

$$\sum_{\nu=1}^{q} \{T_0(r) - \overline{N}(r, a_\nu)\} \leqq 2T_0(r) + o(T_0(r)), \tag{2}$$

where (2) *holds for all r if f is of finite order and* (2) *holds off a set of finite Lebesgue measure if f is of infinite order.*

(It is to be noted that the bound on the remainder term in (2) is not as sharp as is the corresponding bound in Nevanlinna's second fundamental theorem; however the size of the set of exceptional r-values is the same in both instances. The only other difference between the conclusion of the above theorem and the classical theorem is that (2) does not contain the term $N_0(r, 1/f')$, the integrated counting function of the set of points z such that $f'(z) = 0$ and $f(z) \neq a_\nu$, $1 \leqq \nu \leqq q$.)

PROOF. It is sufficient to show that $\int_0^r (L(t)/t)dt = o(T_0(r))$ for the required set of r-values. It follows from the formulae expressing $S(t)$

and $L(t)$ that $L(t) \leqq \pi(2tS'(t))^{1/2}$. Let r_0 be such that $S(r_0)=1$. We make the following computation, using the Schwarz inequality at the third step:

$$\begin{aligned}\int_0^r \frac{L(t)}{t}\,dt &\leqq \pi 2^{1/2}\int_{r_0}^r \frac{(tS'(t))^{1/2}}{t}\,dt + O(1)\\ &= \pi 2^{1/2}\int_{r_0}^r \frac{S'(t)^{1/2}S(t)^{1/2}}{S(t)^{1/2}t^{1/2}}\,dt + O(1)\\ &\leqq \pi 2^{1/2}\left(\int_{r_0}^r \frac{S'(t)}{S(t)}\,dt\right)^{1/2}\left(\int_{r_0}^r \frac{S(t)}{t}\,dt\right)^{1/2} + O(1)\\ &\leqq \pi 2^{1/2}(\log S(r))^{1/2}T_0(r)^{1/2} + O(1).\end{aligned}$$

If f is of finite order, then $\log S(r) \leqq \log T_0(er) \leqq A \log r$ for some $A>0$ and all r greater than some r_1. Thus if $r>r_1$, we have

$$\frac{1}{T_0(r)}\int_0^r \frac{L(t)}{t}\,dt \leqq \left(\frac{2\pi^2 A \log r}{T_0(r)}\right)^{1/2} + \frac{O(1)}{T_0(r)}\,.$$

Since f is transcendental, $\log r/T_0(r) \to 0$ and the result follows.

Now suppose f has infinite order. Define $E_2=\{r: T_0(r)^2<T_0'(r)\}$. Let $E_3=E_2\cap[r_0, \infty)$. Then

$$m(E_3) = \int_{E_3} dt \leqq \int_{r_0}^{\infty} \frac{T_0'(t)}{T_0(t)^2}\,dt = \frac{1}{T_0(r_0)} < \infty\,.$$

Furthermore, for all $r \notin E_2$,

$$\log S(r) = \log rT_0'(r) \leqq \log r + 2\log T_0(r).$$

Thus

$$\lim_{r\to\infty;\, r\notin E_2} \frac{\log S(r)}{T_0(r)} = 0$$

and the result follows.

References

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University of Wisconsin