ON A THEOREM OF AZBELEV AND CALJUK

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The purpose of this note is to give a counterexample to a well-known theorem of Azbelev and Caljuk [1]. For completeness the appropriate definitions and the theorem in question are quoted from [1].

"Consider the equation $L[y]=y'''+p_2(x)y''+p_1(x)y'+p_0(x)y=0 \cdots$. Two adjacent zeros of a solution y (of $L[y]=0$) will be called (i, k)-adjacent zeros of the solution y if the multiplicity of the first is not less than i and that of the latter not less than k.

"Let $t \in [0, \infty)$ be a fixed point. A semiclosed interval $[t, b)$ in which there are no (i, k)-adjacent zeros of solutions $\cdots$ will be called an interval of $(i, k)_t$-nonoscillation. The maximum interval of $(i, k)_t$-nonoscillation for fixed t is denoted by $[t, r_{ik}(t)) \cdots$.

"We will show $\cdots r_{22}(t) = \max\, [r_{12}(t), r_{21}(t)]$."

This last statement is the main result of their paper.

Consider now the equation

$$(3 \sin^2 x \cos^2 x - 2)y''' - 6 \sin x \cos x(\cos^2 x - \sin^2 x)y'' - (9 \sin^2 x \cos^2 x + 14)y' = 0.$$

Note that $3 \sin^2 x \cos^2 x - 2 < 0$. This equation has as a fundamental set of solutions $\sin^2 x \cos x$, $\cos^2 x \sin x$, and 1. It is easily seen that $r_{12}(0) = r_{21}(0) = \pi < r_{22}(0)$.

The error in the proof offered in [1] occurs in drawing the incorrect conclusions from the implicit function theorem in their proof of Lemma 7. It should be noted that their result is however correct in the case that $r_{12}(t) > r_{21}(t)$. Some other conditions under which $r_{22}(t) = \max\, [r_{12}(t), r_{21}(t)]$ are given in the Ph.D. dissertation of Mr. Grant Gustafson.

Reference

1. N. V. Azbelev and Z. B. Caljuk, *On the question of distribution of zeros of solutions of linear differential equations of third order*, Mat. Sb. **51** (93) (1960), 475–486; English transl., Amer. Math. Soc. Transl. (2) **42** (1964), 233–245.

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