WATTS COHOMOLOGY OF FIELD EXTENSIONS[1]

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Let R be a commutative ring and A a commutative R-algebra. In [4] Watts defined a cohomology theory, $H_K^n(A)$, which yields the Čech cohomology of the compact Hausdorff space X in the case when $R=\mathbf{R}$ and $A=C(X)$, the ring of continuous real valued functions on X. The definition of $H_R^n(A)$ was in terms of a specific complex derived from the "additive Amitsur complex." The question of the possible functorial significance of this cohomology theory was raised. As a step in this direction we compute here the Watts cohomology $H_K^n(L)$, where K is a field and L is an arbitrary extension field of K.

We recall the definition of $H_K^n(L)$. The complex $F_K^{\cdot}(L)$ is the additive Amitsur complex [3] with a dimension shift of 1: $F_K^n(L)$ is the $n+1$-fold tensor product of L over K, and the coboundary map $d^n: F_K^n(L)\to F_K^{n+1}(L)$ is given by

$$d^n(x_0\otimes\cdots\otimes x_n)=\sum_{i=0}^{n+1}(-1)^i x_0\otimes\cdots\otimes x_{i-1}\otimes 1\otimes x_i\otimes\cdots\otimes x_n.$$

The homology of this complex is easily found.

PROPOSITION 1. *The complex $F_K^{\cdot}(L)$ has zero homology except in dimension zero, where $H^0(F_K^{\cdot}(L))\cong K$.*

PROOF. It is known [3, Lemma 4.1] that the complex $0\to K\to F_K^0(L)\to F_K^1(L)\cdots$ is acyclic.

Let μ_n: $F_K^n(L)\to L$ by $\mu_n(x_0\otimes\cdots\otimes x_n)=x_0\cdots x_n$. The subcomplex $N_K^{\cdot}(L)$ is given by

$$N_K^n(L)=\{x\in F_K^n(L)\mid \exists y\in F_K^n(L)\quad\text{with}\quad \mu_n(y)\neq 0\quad\text{and}\quad xy=0\}$$

(the definition is simplified here by the fact that L is a field). Note that $N_K^n(L)\subseteq\ker\mu$. The Watts cohomology $H_K^{\cdot}(L)$ is then defined to be the homology of the quotient complex $C_K^{\cdot}(L)=F_K^{\cdot}(L)/N_K^{\cdot}(L)$. Let $\pi_n: F_K^n(L)\to C_K^n(L)$ denote the standard map.

Let L_s be the separable closure of K in L. We shall prove the following

THEOREM. *The complexes $C_K^{\cdot}(L)$ and $F_{L_s}^{\cdot}(L)$ are canonically isomorphic.*

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The following corollary is then immediate by Proposition 1.

COROLLARY. *Watts cohomology for field extensions is given by* $H_K^0(L) \cong L_s$ *and* $H_K^n(L) = 0$ *for* $n > 0$.

We now establish the theorem.

PROPOSITION 2. *Let L be separable algebraic over K, and let $x \in F_K^n(L)$ with $\mu_n(x) = 0$. Then there is a $y \in F_K^n(L)$ with $\mu_n(y) \neq 0$ and $xy = 0$.*

PROOF. It suffices to consider the case where L is a finite extension of K. But then $F_K^n(L)$ is a semisimple ring and every ideal is a direct factor. Hence $F_K^n(L) \cong L' \times \ker(\mu_n)$, where L' is a field and μ_n maps L' isomorphically onto L.

COROLLARY. *If L is separable algebraic over K, $N_K^n(L) = \ker(\mu_n)$, and there is a canonical isomorphism $\beta_n : C_K^n \to L$ such that $\beta_n \circ \pi_n = \mu_n$.*

PROPOSITION 3. *Let K be separably closed in L (i.e. $L_s = K$) and let A be a commutative K-algebra in which every zero divisor is nilpotent. Then every zero divisor in $A \otimes_K L$ is nilpotent.*

PROOF. We may work within a finitely generated subalgebra of A, and hence assume A is Noetherian. Let N be the ideal of nilpotents in A. Then (0) is a primary ideal in A and N is its associated prime. It then follows [1, Chapter IV, §2.6, Theorem 2] (with $E = A$, $F = B = A \otimes_K L$) that the associated prime ideals of (0) in $A \otimes_K L$ coincide with the associated prime ideals of the ideal $N \otimes_K L$ in $A \otimes_K L$. But by [2, Chapter IV, Theorem 24] every zero-divisor in $A/N \otimes_K L$ is nilpotent and hence $N \otimes_K L$ is a primary ideal.

COROLLARY. *If K is separably closed in L, then every zero divisor in $F_K^n(L)$ is nilpotent, for all n, and $F_K^n(L) = C_K^n(L)$.*

Now let $\theta_n : F_K^n(L) \to F_{L_s}^n(L)$ by $\theta_n(x_0 \otimes \cdots \otimes x_n) = x_0 \otimes \cdots \otimes x_n$. It is clear that θ_n is surjective and that $\theta = \{\theta_n\}$ is a map of complexes, $\theta : F_K^\cdot(L) \to F_{L_s}^\cdot(L)$. Let $x \in N_K^n(L)$, with y satisfying $xy = 0$, $\mu_n(y) \neq 0$. Then $\theta_n(x)\theta_n(y) = 0$ and $\mu_n(\theta_n(y)) = \mu_n(y) \neq 0$, so $\theta_n(x) \in N_{L_s}^n(L) = (0)$. Hence θ induces a surjective map of complexes $\tau: C_K^\cdot(L) \to C_{L_s}^\cdot(L) = F_{L_s}^\cdot(L)$. To complete the proof of the theorem we will construct an inverse to τ.

Fix an integer n, and let A denote the ring $F_K^n(L)$, B the subring $F_K^n(L_s)$. If M is an A-module and $\rho: A \to M$ an A-linear map, then $\ker(\rho) \supset A(\ker \rho|_B)$ and hence ρ factors as $A = A \otimes_B B \to A \otimes_B \rho(B) \to M$. We apply this method to the two maps $\theta_n: A \to F_{L_s}^n(L)$, and $\pi_n: A \to C_K^n(L)$.

Proposition 4. *The induced map* $A\otimes_B\theta_n(B)=F^n_K(L)\otimes_{F^n_K(L_s)}L_s \to F^n_{L_s}(L)$ *is an isomorphism.*

Proof. First note that $\theta_n(B)$ is canonically isomorphic to L_s. We construct an inverse. Let $\omega\colon L\times\cdots\times L\to A\otimes_B\theta_n(B)$ by $\omega(x_0,\cdots,x_n) = x_0\otimes\cdots\otimes x_n\otimes 1$. If $y\in L_s$, $x_0\otimes\cdots\otimes yx_i\otimes\cdots\otimes x_n\otimes 1 = x_0\otimes\cdots\otimes x_i\otimes\cdots\otimes x_n\otimes y$. Hence ω is L_s-multilinear and induces $F^n_{L_s}(L)\to A\otimes_B\theta_n(B)$.

Consider the following diagram:

$$\begin{tikzcd} A = A \otimes_B B \arrow[r] \arrow[d] & A \otimes_B \theta_n(B) \arrow[r] \arrow[dl, dashed] & F^n_{L_s}(L) \\ A \otimes_B C^n_K(L_s) \arrow[d] & & \\ C^n_K(L) & & \end{tikzcd}$$

Using Propositions 3 and 4, $\theta_n(B)\cong L_s\cong C^n_K(L_s)$ and hence we obtain, by Proposition 4,

$$F^n_{L_s}(L)\to A\otimes_B\theta_n(B)\to A\otimes_B C^n_K(L_s)\to C^n_K(L),$$

and the composite map is easily seen to be inverse to τ_n.

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