THE ORDER OF THE ANTIPODE OF A HOPF ALGEBRA

RICHARD GUSTAVUS LARSON[1]

The purpose of this note is to show that the order of the antipode of a Hopf algebra is not necessarily 2, but can be any positive even integer or infinite.

A *coalgebra* over a field K is a vector space C over K together with maps $\delta: C \to C \otimes C$ and $\epsilon: C \to K$ satisfying

$$(\delta \otimes 1)\delta = (1 \otimes \delta)\delta \quad \text{and} \quad (\epsilon \otimes 1)\delta = (1 \otimes \epsilon)\delta = 1.$$

The coalgebra is *cocommutative* if $t\delta = \delta$, where $t: C \otimes C \to C \otimes C$ is defined by $t(c \otimes d) = d \otimes c$. A *Hopf algebra* over K is an associative algebra H with identity, together with identity-preserving algebra morphisms $\delta: H \to H \otimes H$ and $\epsilon: H \to K$ which give the underlying vector space a coalgebra structure. If H is an associative algebra, we will denote by $\mu: H \otimes H \to H$ the map defined by $\mu(h \otimes k) = hk$, and by $\eta: K \to H$ the map defined by $\eta(a) = a1$. An *antipode* for the Hopf algebra H is a map $\gamma: H \to H$ satisfying

$$\mu(\gamma \otimes 1)\delta = \mu(1 \otimes \gamma)\delta = \eta\epsilon.$$

If there exists an antipode γ for the Hopf algebra H, then it is unique, and is a Hopf algebra antiendomorphism. If H is either commutative or cocommutative, then $\gamma^2 = 1$. (See §8 of [1] for proofs of these facts.) The *order* of the antipode γ is the smallest positive integer n such that $\gamma^n = 1$, if such an integer exists, and is infinite otherwise. Since γ is an antiendomorphism, if γ has finite order it must have even order, unless H is both commutative and cocommutative, in which case γ may have order 1.

THEOREM. *If n is a positive even integer or infinite, there exists a Hopf algebra over* $\mathbf{R}$ *which has an antipode of order n.*

Free Hopf algebras over coalgebras. If X is a vector space over K, we denote by $T(X)$ the tensor algebra of X. Let C with the maps δ, ϵ be a coalgebra. The map $C \to T(C) \otimes T(C)$ given by $c \to \delta(c)$ (where we are identifying $C \otimes C$ with a subspace of $T(C) \otimes T(C)$ by means of the usual identification of C with a subspace of $T(C)$) induces an algebra morphism $\delta_T: T(C) \to T(C) \otimes T(C)$. Also the map $\epsilon: C \to K$ induces an

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algebra morphism $\epsilon_T\colon T(C)\to K$. It is easily seen that $T(C)$ together with the maps δ_T and ϵ_T is a Hopf algebra, called the *free Hopf algebra on C*.

An antiautomorphism of the coalgebra C is a bijective map $\zeta\colon C\to C$ satisfying

$$\delta\zeta = (\zeta \otimes \zeta)t\delta \quad \text{and} \quad \epsilon\zeta = \epsilon.$$

Given an antiautomorphism of C, consider the ideal $I\subset T(C)$ generated by all elements of the form

$$\epsilon_T(c)1 - \mu_T(\zeta \otimes 1)\delta_T(c) \quad \text{and} \quad \epsilon_T(c)1 - \mu_T(1 \otimes \zeta)\delta_T(c),$$

where $c\in C\subset T(C)$. It is easily checked that $\delta(I)\subset T(C)\otimes I+I\otimes T(C)$ and that $\epsilon(I)=0$. This implies that $H(C;\zeta)=T(C)/I$ together with the maps δ_H, ϵ_H induced by δ_T, ϵ_T is a Hopf algebra. If κ is the unique antiendomorphism of $T(C)$ satisfying $\kappa|C=\zeta$, then $\kappa(I)\subset I$, so κ induces a map $\gamma_H\colon H(C;\zeta)\to H(C;\zeta)$.

The facts that γ_H is an algebra antiendomorphism and that

$$\mu_H(\gamma_H \otimes 1)\delta_H(c + I) = \mu_H(1 \otimes \gamma_H)\delta_H(c + I) = \eta_H\epsilon_H(c + I)$$

for all $c\in C$ imply that γ_H is an antipode for the Hopf algebra $H(C;\zeta)$. The Hopf algebra $H(C;\zeta)$ is called the *free Hopf algebra with antipode on C and ζ*. Denote by $\pi\colon C\to H(C;\zeta)$ the composition of the maps $C\to T(C)\to H(C;\zeta)$.

Warning. The map $\pi\colon C\to H(C;\zeta)$ need not be injective.

The following Proposition is immediate:

PROPOSITION. *Let C be a coalgebra, and let ζ be an antiautomorphism of C. Then there exist a Hopf algebra $H(C;\zeta)$ with antipode γ, and a morphism of coalgebras $\pi\colon C\to H(C;\zeta)$ with $\pi\zeta=\gamma\pi$, such that for every Hopf algebra H with antipode γ' and every coalgebra morphism $f\colon C\to H$ satisfying $\gamma' f=f\zeta$, there exists a unique Hopf algebra morphism $g\colon H(C;\zeta)\to H$ with $g\pi=f$.*

Construction of the example. We prove the Theorem as follows: we construct a coalgebra C with antiautomorphism ζ of order n such that $\pi\colon C\to H(C;\zeta)$ is injective. Then $\gamma|\pi(C)=\pi\zeta\pi^{-1}|\pi(C)$ is of order n, so that γ is of order at least n. On the other hand, it is clear that γ is of order at most n.

To construct C and ζ we construct a finite dimensional algebra A over $\mathbf{R}$ with an antiautomorphism σ of order n, and define $C=A^*=\hom(A, \mathbf{R})$, $\delta=\mu^t$, $\epsilon=\eta^t$, and $\zeta=\sigma^t$. If A has a basis $\{a_i\}$ and a multiplication table $a_ia_j=\sum m_{ijk}a_k$, then δ is given explicitly by $\delta(a_k^*)=\sum m_{ijk}a_i^*\otimes a_j^*$, where $\{a_i^*\}$ is the basis of C dual to the basis

$\{a_i\}$. If $1=\sum e_i a_i$, then $\epsilon(a_i^*)=e_i$. If $\sigma(a_i)=\sum s_{ij}a_j$, then $\zeta(a_j^*)=\sum s_{ij}a_i^*$. Thus, in this case $H(C;\zeta)$ can be described as the associative algebra generated by $\{a_i^*\}$, subject to the relations

$$e_i = \sum m_{lki}s_{jl}\overset{*}{a}_j\overset{*}{a}_k \quad \text{and} \quad e_i = \sum m_{jli}s_{kl}\overset{*}{a}_j\overset{*}{a}_k.$$

To prove that $\pi: C\to H(C;\zeta)$ is injective, it is sufficient to find a representation $\rho: H(C,\zeta)\to \hom(V, V)$ such that $\{\rho(a_i^*)\}$ is linearly independent.

Let A be the algebra of all 2×2 matrices over $\boldsymbol{R}$. If n is a positive even integer, let $\theta=2\pi/n$. If n is infinite, let $\theta=\alpha\pi$, where α is any irrational number. Let

$$U = \begin{vmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{vmatrix}.$$

Define $\sigma: A\to A$ by $\sigma(T)=U^{-1}T^tU$, for all $T\in A$. The matrices

$$C' = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \quad S' = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix},$$

$$X' = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \quad Y' = \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix},$$

are a basis for A. With respect to this basis, $\sigma(C')=C'$, $\sigma(S')=-S'$, $\sigma(X')=\cos\theta X'+\sin\theta Y'$, and $\sigma(Y')=-\sin\theta X'+\cos\theta Y'$. It is clear that the antiautomorphism σ is of order n.

Let C, S, X, and Y be the basis of $C=A^*$ dual to the given basis of A. We apply the discussion in the second paragraph of this section to show that $H(C;\zeta)$ is the algebra generated by C, S, X, and Y, subject to the relations

(1) $1=C^2+S^2+X\gamma(X)+Y\gamma(Y)$,
(2) $1=C^2+S^2+\gamma(X)X+\gamma(Y)Y$,
(3) $0=-CS+SC+X\gamma(Y)-Y\gamma(X)$,
(4) $0=CS-SC+\gamma(X)Y-\gamma(Y)X$,
(5) $0=C\gamma(X)+S\gamma(Y)+XC+YS$,
(6) $0=CX-SY+\gamma(X)C-\gamma(Y)S$,
(7) $0=C\gamma(Y)-S\gamma(X)-XS+YC$,
(8) $0=CY+SX+\gamma(X)S+\gamma(Y)C$,

where $\gamma(X)=\cos\theta X-\sin\theta Y$ and $\gamma(Y)=\sin\theta X+\cos\theta Y$.

Some straightforward calculations show that equations (1)–(4) are equivalent to

(9) $XY=YX$,
(10) $1=C^2+S^2+\cos\theta(X^2+Y^2)$,
(11) $0=CS-SC-\sin\theta(X^2+Y^2)$,

and that the equations (5)–(8) are equivalent to

(12) $0=CX+\cos\theta XC-\sin\theta XS$,

(13) $0=SX+\sin\theta XC+\cos\theta XS$,

(14) $0=CY+\cos\theta YC-\sin\theta YS$,

(15) $0=SY+\sin\theta YC+\cos\theta YS$.

Therefore $H(C;\zeta)$ is the algebra generated by C, S, X, and Y, subject to the slightly less formidable relations (9)–(15).

We now define the representation ρ. Consider the representation of $T(C)$ on a three dimensional vector space defined by

$$C\to\begin{vmatrix} 1 & 0 & 0 \\ 0 & -\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}, \qquad S\to\begin{vmatrix} 0 & 0 & 0 \\ 0 & -\sin\theta & 0 \\ 0 & 0 & 0 \end{vmatrix},$$

$$X\to\begin{vmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}, \qquad Y\to\begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{vmatrix};$$

Routine calculations show that this representation preserves the relations (9)–(15) and so induces a representation ρ of $H(C;\zeta)$ on this vector space. Clearly $\rho(C)$, $\rho(S)$, $\rho(X)$ and $\rho(Y)$ are linearly independent if $n>2$. Therefore the antipode of $H(C;\zeta)$ has order n. Q.E.D.

Reference

1. J. W. Milnor and J. C. Moore, *On the structure of Hopf algebras*, Ann. of Math. **81** (1965), 211–264.

University of Illinois at Chicago Circle