# MODULARITY IN TOPOLOGICAL LATTICES

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The purpose of this note is to establish that for topological lattices of suitably small breadth, connectedness implies modularity without an exploitation of compactness. $L$ will denote a topological lattice, that is a Hausdorff topological space with continuous binary operations $\vee$ and $\wedge$ for which $(L, \vee, \wedge)$ is a lattice. For a more explicit presentation and some related machinery, see the paper of E. Dyer and A. Shields [5]. Of specific need, $L$ is a modular lattice iff for every $a, b, c \in L$, $b \leqq a$ implies $a \wedge (b \vee c) = b \vee (a \wedge c)$. And if $n$ is a positive integer, the breadth of $L$ is less than $n$ means that for every $x_1, x_2, \cdots, x_n \in L$, there exists $j$ such that $x_j \leqq \bigvee_{i \neq j} x_i$. Note that the breadth formulation given here is dual to the equivalent phrasing used in [5]. Also for $a \leqq b$ in $L$, define $[a, b] = \{x \in L \mid a \leqq x \leqq b\}$.

THEOREM 1. *If $L$ is a connected topological lattice and the breadth of $L$ is less than three, then $L$ is modular.*

PROOF. It suffices to show that $L$ does not contain a nonmodular five lattice [4]. Thus let $a, b, c \in L$ such that $b \leqq a$, $a \wedge c = b \wedge c$, and $a \vee c = b \vee c$. Under the hypotheses of the theorem it will be established that $b = a$, and therefore $L$ cannot contain a nonmodular five lattice.

For $x$ such that $a \leqq x \leqq a \vee c$ and $a \wedge c \neq x \wedge c$, let

$$A = \{y \in [c, c \vee a] \mid a \wedge y \leqq b \vee (x \wedge c)\}.$$

Clearly $A \cup \{c \vee a\} \subset [c, c \vee a]$. Now if $c \leqq y < c \vee a$, the breadth condition implies (1) $a \wedge y \leqq b \vee (x \wedge c)$, (2) $b \leqq (a \wedge y) \vee (x \wedge c)$, or (3) $x \wedge c \leqq b \vee (a \wedge y)$. But (2) implies $b \leqq (a \wedge y) \vee c \leqq y$ and hence $a \vee c = b \vee c \leqq y$, contrary to $y < a \vee c$. Also (3) implies $x \wedge c \leqq a$ and therefore $x \wedge c = a \wedge c$, contrary to assumption. Thus if $c \leqq y < c \vee a$, then $y \in A$ and $A \cup \{c \vee a\} = [c, c \vee a]$. But $A$ and $\{c \vee v\}$ are closed nonempty sets whose union is a connected set since $L$ is connected. Thus $c \vee a \in A$ and $a = a \wedge (c \vee a) \leqq b \vee (x \wedge c)$. Therefore $[a, a \vee c] = B \cup C$ where

$$B = \{x \in [a, a \vee c] \mid a \wedge c = x \wedge c\}$$

and

$$C = \{x \in [a, a \vee c] \mid a \leqq b \vee (x \wedge c)\}.$$

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Immediately $B$ and $C$ are nonempty closed sets and since $[a, a\vee c]$ is connected, $B\cap C\neq\varnothing$. Therefore there exist $d$ such that $a\wedge c=d\wedge c$ and $a\leqq b\vee(d\wedge c)$, and $a\leqq b\vee(d\wedge c)=b\vee(a\wedge c)=b\vee(b\wedge c)=b$. Thus $a=b$.

THEOREM 2. *If $L$ is a locally compact connected topological lattice of codimension less than three, $L$ is modular.*

PROOF. Lawson [7] has shown that a locally compact connected topological lattice of codimension $n$ has breadth less than or equal to $n$. This generalizes the result of L. W. Anderson and L. E. Ward, Jr. [1] Thus the theorem follows from Theorem 1.

REMARK. In an earlier paper [6] the author gave an example of a compact connected topological lattice of dimension three which is nonmodular. This example is seen to have breadth three also, thus in some respects the result above is the best possible.

## REFERENCES

1. L. W. Anderson and L. E. Ward, Jr., *A structure theorem for topological lattices*, Proc. Glasgow Math. Assoc. 5 (1961), 1–3.

2. Lee W. Anderson, *On breadth and codimension of a topological lattice*, Pacific J. Math. 9 (1959), 327–333.

3. ———, *One dimensional topological lattices*, Proc. Amer. Math. Soc. 10 (1959), 715–720.

4. Garrett Birkhoff, *Lattice theory*, Amer. Math. Soc. Colloq. Publ., Vol. 25, Amer. Math. Soc., Providence, R. I., 1948.

5. Eldon Dyer and Allen Shields, *Connectivity of topological lattices*, Pacific J. Math. 9 (1959), 443–448.

6. Don E. Edmondson, *A nonmodular compact connected topological lattice*, Proc. Amer. Math. Soc. 7 (1956), 1157–1158.

7. J. D. Lawson, *Vietoris mappings and embeddings of topological semi-lattices*, Dissertation, Univ. of Tennessee, Knoxville, 1967.

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