A NOTE ON "NTH ROOTS OF OPERATORS"

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M. R. Embry has proved the following theorem [1]:

THEOREM [E]. "Let A and E be bounded operators on a Banach space, and suppose $\sigma(A)\cap\sigma(e^{2\pi i\, k/n}A)=\varnothing$ for $k=1, 2, \cdots, n-1$. Then if E commutes with A^n, it commutes with A."

This result can be obtained as a corollary to the following theorem:

Notation (*as in* [2]). Let T be a bounded operator on a Banach space. Let $\mathfrak{F}(T)=$ the functions holomorphic on some neighborhood of $\sigma(T)$. Let $f\in\mathfrak{F}(T)$.

As usual,

$$f(T) = (2\pi i)^{-1}\int_C f(\lambda)(\lambda - T)^{-1}d\lambda.$$

THEOREM. *In order that T be representable as a function of $f(T)$ it is sufficient that* (1) *f is* 1-1 *on $\sigma(T)$, and* (2) *$f'\neq 0$ on $\sigma(T)$. Consequently if E is a bounded operator and commutes with $f(T)$ then it commutes with T.*

PROOF. (1) and (2) ensure that f is 1-1 on a neighborhood of $\sigma(T)$, for if not there exist $\{z_n\}$, $\{z_n'\}$, $z_n\neq z_n'$, such that $f(z_n)=f(z_n')$ and these may be chosen so that $z_n\to z_0$, $z_n'\to z_0'$, $z_0\in\sigma(T)$, $z_0'\in\sigma(T)$. Now $z_0=z_0'$, else (1) is violated. But then

$$0 = \frac{f(z_n) - f(z_n')}{z_n - z_n'} \to f'(z_0)$$

which contradicts (2). If f is 1-1 on a neighborhood of $\sigma(T)$, then $f^{-1}\in\mathfrak{F}(f(T))$, and Theorem 12 [2, p. 570] yields $T=f^{-1}(f(T))$.

If f fails to satisfy (1) or (2), an E and T for which E commutes with $f(T)$ but not with T can be found in 2×2 matrices.

Note that z^n satisfies (1) and (2) precisely if the condition in Theorem E is satisfied.

REFERENCES

1. M. R. Embry, *Nth roots of operators*, Proc. Amer. Math. Soc. 19 (1968), 63–68.

2. N. Dunford and J. T. Schwartz, *Linear operators*. I, Interscience, New York, 1958.

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