# ON COHOMOLOGY GROUPS OF BANACH ALGEBRAS

WEI-LUNG TING[1]

In this note we will study the cohomology groups of some special classes of Banach algebras, and show that there are some relationships between the cohomology groups of a Banach algebra (for definition see [2]) and the space of maximal ideals of that Banach algebra We assume that the Banach algebras considered here are the Banach algebras over the field of complex numbers $C$.

Let $\Lambda=\{1; A\}$ be a Banach algebra which is generated by one element $A$. Suppose there is a continuous algebra homomorphism $\chi: \Lambda\to C$ with $\chi(A)=\tau$; then $C$ can be regarded as a two sided Banach $\Lambda$-module with $\lambda c=\chi(\lambda)c$, for $c\in C$ and $\lambda\in\Lambda$. Now we want to calculate the one dimensional cohomology group of the Banach algebras, which have one generator, with coefficients in $C$.

Let $\Lambda=\{1, A\}$ and $\chi: \Lambda\to C$ be the Banach algebra and continuous algebra homomorphism defined above and suppose that $\chi(A)=\tau$. Then $\mathfrak{M}_\Lambda=\chi^{-1}(0)$ is a maximal ideal of $\Lambda$ and $\Lambda/\mathfrak{M}_\Lambda\cong C$. Let $Z^1$ denote the set of all 1-cocycles, then $f\in Z^1$ if and only if $f: \Lambda\to C$ is a bounded linear function and satisfying the identity

$$f(\lambda_1\lambda_2) = \chi(\lambda_1)f(\lambda_2) + \chi(\lambda_2)f(\lambda_1).$$

If $\lambda_1, \lambda_2\in\mathfrak{M}_\Lambda$ then $f(\lambda_1\lambda_2)=0$; this implies that $\mathfrak{M}_\Lambda^2\subset f^{-1}(0)$. If the closed subspace of $\Lambda$, which is generated by $\mathfrak{M}_\Lambda^2$ and $C$, is denoted by $T$; then since $f(1)=0$, one finds that $T\subset f^{-1}(0)$. Hence $f$ induces a continuous linear function $\hat{f}: \Lambda/T\to C$. Now if $\Lambda/T\cong 0$ then $H^1(\Lambda, C)=0$.

PROPOSITION 1. *Suppose $\Lambda=\{1; A\}$ is a Banach algebra with the generator $A$ while $\chi: \Lambda\to C$ is a continuous algebra homomorphism with $\chi(A)=\tau$. If $C$ is considered as a two sided Banach $\Lambda$-module with $\lambda c=\chi(\lambda)c$, then $H^1(\Lambda; C)\cong C$ if and only if, for any sequence of polynomials $\{P_n(A)\}$ where $P_n(A)=a_0^n+a_1^nA+\cdots+a_{m_n}^nA^{m_n}$, if $\{P_n(A)\}$ converges to zero in $\Lambda$, then the sequence of its formal derivatives at $\tau\{P_n'(\tau)\}$, where $P_n'(\tau)=a_1^n+\cdots+m_na_{m_n}^n\tau^{m_n-1}$, converges to zero in $C$.*

PROOF. If $H^1(\Lambda; C)\cong C$ then there is a bounded 1-cocycle $f$ such that $f(A)\neq 0$. Now given sequence of polynomials $\{P_n(A)\}$ that

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converges to zero in $\Lambda$ the sequence $\{f(P_n(A))\}_{n=1,2\cdots}$ will converge to zero in $C$. But we have $f(P_n(A))=f(A)P_n'(\tau)$. Hence the sequence $\{P_n'(\tau)\}$ converges to zero in $C$.

Conversely, suppose that for any sequence of polynomials $\{P_n(A)\}$ that converges to zero in $\Lambda$, the sequence $\{P_n'(\tau)\}$ converges to zero in $C$. Let us define the continuous linear function $f\colon \Lambda\to C$ by $f(P_n(A)) = P_n'(\tau)$. We can show that $f$ is a 1-cocycle, and $f\neq 0$. Hence

$$H^1(\Lambda, C)\cong C.$$

Now, we shall study a certain Banach algebra whose elements are functions defined and continuous on some closed set $E\subset C$, with a norm given by the maximum modules of function. Let $\tilde{\Lambda}$ denote the Banach algebra generated by the function $z$ on a bounded closed set $E\subset C$ ($z$ being the identity map of $E$). In order to study $H^1(\tilde{\Lambda}, C)$ we need the following result.

LEMMA 2 [4, p. 310]. *In order that the function $f(z)$, defined on a closed set $E\subset C$, can be expanded in a series of polynomials in $z$ which converges uniformly to $f(z)$ on $E$, it is necessary and sufficient that the complement of $E$ should consist of one region which contains the point of infinity, and that $f(z)$ should be continuous on $E$ and analytic in every interior point of the set $E$.*

According to a theorem (in [1, p. 72]), a bounded closed set $E$ in the complex plane, which is homeomorphic to the space of maximal ideals of some $\Lambda=\{1, z\}$, may fall into any one of the two following types:

(i) The set $E$ does not have interior points and does not bound any domain (in the sense that every point $\xi$ not in $E$ can be joined to $\infty$ by a line passing outside $E$).

(ii) The set $E$ has interior points, but does not bound any domain.

When $E$ is of type (i), then by Lemma 2, $\tilde{\Lambda}$ is the Banach algebra of all continuous functions defined on $E$. Let $\tau$ be any element of $E$, then the function $\chi_\tau\colon \tilde{\Lambda}\to C$ defined by $\chi_\tau(\lambda)=\lambda(\tau)$, for $\lambda\in\tilde{\Lambda}$, is a continuous algebra homomorphism and $\chi_\tau(z)=\tau$. When we regard $C$ as a two sided Banach $\Lambda$-module by $\chi_\tau$, then by [2] we have

$$H^1(\tilde{\Lambda}, C) = 0.$$

When $E$ is of type (ii), then by Lemma 2 $\tilde{\Lambda}$ is the Banach algebra of all continuous functions on $E$ that are analytic in every interior point of the set $E$.

THEOREM 3. *When $E$ is of type* (ii), *then there exists a point $\tau\in E$ such that the function $\chi_\tau$ defined by $\chi(\lambda)=\lambda(\tau)$, for $\lambda\in\tilde{\Lambda}$, is a continuous* *algebra homomorphism and if we regard $C$ as a two sided Banach $\Lambda$-module such that, for $c \in C$ and $\lambda \in \Lambda$, $\lambda c = \chi_\tau(\lambda) \cdot c$, then $H^1(\tilde{\Lambda}, C) \cong C$.*

PROOF. Let $\tau \in E$ be an interior point of $E$. Since every element of $\tilde{\Lambda}$ is analytic at $\tau$ the function $f: \tilde{\Lambda} \to C$ defined by $f(\lambda) = \lambda'(\tau)$ is a 1-cocycle. To prove $f$ is continuous we define a function $\tilde{\lambda}(\gamma)$, for $\gamma \in E$ by setting

$$\tilde{\lambda}(\gamma) = \frac{\lambda(\gamma) - \lambda(\tau)}{\gamma - \tau} \qquad \text{if } \gamma \neq \tau$$

and

$$\tilde{\lambda} = \lambda'(\tau).$$

Then $\tilde{\lambda}$ is a function which is continuous on $E$ and analytic in every interior point of $E$. By a theorem in [1, p. 75], $\tilde{\lambda}$ assumes its maximum value on the topological boundary $\Gamma$ of $E$. Since $\tau$ is an interior point of $E$, we have $\min_{\gamma \in \Gamma} |\gamma - \tau| = \gamma_0 \neq 0$. Hence $|\tilde{\lambda}(\tau)| = |\lambda'(\tau)| < 2/\gamma_0 |\lambda|$. This shows that $f$ is a bounded 1-cocycle, and $H^1(\tilde{\Lambda}, C) \cong C$.

From above results and a result in [1, p. 73], we have the following theorem.

THEOREM 4. *A necessary and sufficient condition that there exists a continuous algebra homomorphism $\chi: \tilde{\Lambda} \to C$ with $|\chi(\lambda)| \leqq |\lambda|$ which permits us to regard $C$ as a two sided Banach $\tilde{\Lambda}$-module with $\lambda c = \chi(\lambda) c$ and such that $H^1(\tilde{\Lambda}, C) \cong C$ is that $E$ has interior points.*

REMARK. Even if $E$ has interior points there are also some points $\tau \in E$ such that, $H^1(\tilde{\Lambda}, C) = 0$ where $C$ is considered as a two sided Banach $\Lambda$-module by means of the continuous algebra homomorphism $\chi: \tilde{\Lambda} \to C$ defined by $\chi_\tau(\lambda) = \lambda(\tau)$ for $\lambda \in \tilde{\Lambda}$.

THEOREM 5. *Let $\Lambda$ be any Banach algebra with one generator $A$; and let us assume that, whenever $C$ becomes a Banach $\Lambda$-module by a continuous algebra homomorphism $\chi: \Lambda \to C$ with $|\chi| \leqq 1$, then $H^1(\Lambda, C) = 0$. Then the space $\mathfrak{M}(\Lambda)$ of maximal ideal of $\Lambda$, which is homeomorphic to a closed bounded subset of the complex plane, has no interior points.*

PROOF. Suppose the closed and bounded subset $E$ of the complex plane, which is homeomorphic to $\mathfrak{M}(\Lambda)$, has interior points. Let $\tilde{\Lambda}$ denote the Banach algebra generated by the generator $z$ on $E$ ($z$ being the identity map of $E$); then there is a continuous algebra homomorphism $\eta: \Lambda \to \tilde{\Lambda}$ such that $|\eta| \leqq 1$ and furthermore the image of $\eta$ is dense in $\tilde{\Lambda}$ and $\eta(A) = z$. Since $E$ has interior points, there exists a point $\tau \in E$ such that $C$ becomes a Banach $\Lambda$-module by the mapping $\chi_\tau: \tilde{\Lambda} \rightarrow C$ and $H^1(\tilde{\Lambda}, C) \cong C$. Hence there is a continuous 1-cocycle $f: \tilde{\Lambda} \rightarrow C$ such that, $f \neq 0$. Now if we put $\chi = \chi_\tau \eta$ and $\bar{f} = f\eta$, then $|\chi| \leqq 1$ and $\bar{f}$ is a continuous 1-cocycle of $\Lambda$. Moreover $\bar{f} \neq 0$, for $\bar{f} = 0$ would imply $f = 0$. We have thus proved that $C$ becomes a Banach $\Lambda$-module by $\chi$ and that $H^1(\Lambda, C) \neq 0$. This is a contradiction and hence $E$ has no interior points.

## References


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University of Montreal