

TRANSLATES ARE ALWAYS DENSE ON THE HALF LINE

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Let $f(x)$ be a nontrivial function in $L^1(-\infty, \infty)$. The celebrated theorem of Wiener tells us exactly when the linear combinations of the translates of this function are dense in $L^1(-\infty, \infty)$. [Namely, when the Fourier transform of f has no zeros.] Suppose we ask, however, when these translates are dense in $L^1(0, \infty)$. The surprising answer is: *always*.

Similar remarks hold for the other L^p classes and also for some sets slightly larger than the half line but we omit the details. We prove only the

THEOREM. *If $f(x)$ is any nontrivial function in $L^1(-\infty, \infty)$ then the translates of $f(x)$ span all of $L^1(0, \infty)$.*

The first step of the proof is Wiener's "localization" lemma, see [1]. Writing $\hat{f}(x)$ for the Fourier transform of $f(x)$ and introducing the usual "triangle" function $T_{\xi, \delta}(x) = (\delta - |x - \xi|)_+$ we have

LEMMA 1. *If $\hat{f}(x)$ has no zeros in $[a, b]$ then for sufficiently small $\delta > 0$ and all ξ in $[a + \delta, b - \delta]$ the quotient $T_{\xi, \delta}(x)/\hat{f}(x)$ is the Fourier transform of an $L^1(-\infty, \infty)$ function.*

We also need the following simple lemma

LEMMA 2. *If $F(x)$ is $L^1(0, \infty)$ and non-0 a.e. and if ξ varies through a set of positive measure then the collection of functions $\{F(x)e^{i\xi x}\}$ spans $L^1(0, \infty)$.*

PROOF. Suppose $G(x) \in L^\infty(0, \infty)$ were orthogonal to each of the $F(x)e^{i\xi x}$, that is that $\int_0^\infty G(x)F(x)e^{i\xi x}dx = 0$ for all such ξ . The function defined by $\phi(z) = \int_0^\infty G(x)F(x)e^{izx}dx$ is clearly analytic in $\text{Im } z > 0$ and continuous in $\text{Im } z \geq 0$. Vanishing on a set of positive measure is impossible for the boundary values of such an analytic function unless the function vanished identically. Thus $\phi(z) \equiv 0$ and so $F(x)G(x) = 0$ a.e., and so $G(x) = 0$ a.e. This proves that our collection does indeed span. Our theorem now follows easily. By Lemma 1 we have $T_{\xi, \delta}(x) = \hat{f}(x)\hat{g}(x)$ for some $g \in L^1(-\infty, \infty)$ and so, taking inverse transforms, we have

$$e^{i\xi x} \frac{\sin^2 \delta x}{\pi x^2} = \int_{-\infty}^{\infty} f(x+t)g(-t)dt.$$

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Since the right-hand side is clearly spanned by translates of f we conclude that all the functions $e^{iz}F(x)$ are so spanned for $\xi \in [a+\delta, b-\delta]$, $F(x) = \sin^2 \delta x / \pi x^2$. In turn these functions span $L^1(0, \infty)$ by Lemma 2 and the proof is complete.

REFERENCE

1. G. H. Hardy, *Divergent series*, Clarendon Press, Oxford, 1949, pp. 290–291.

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