RIEMANN SURFACES WITH AUTOMORPHISM GROUPS ADMITTING PARTITIONS[1]

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A finite group G_0, is said to admit a *partition* if there is a set of subgroups G_i, $i=1, 2, \cdots, s$, $s \geqq 2$, so that (1) $G_0 = \bigcup_{i=1}^s G_i$ and (2) if $i \neq j$ and $i, j > 0$, then $G_i \cap G_j = \langle e \rangle$ where e is the identity in G_0.[2] Such groups are well known.[3] The examples considered in this note are abelian groups in which every element has order two, and dihedral groups.

Let W be a closed Riemann surface. A conformal self-map of W will be called an *automorphism*. If G is a finite group of automorphisms of W, then the orbit space, W/G, is naturally a Riemann surface and the natural projection $W \to W/G$ represents W as an n-sheeted covering of W/G where n is the order of G.

The formula embodied in the following lemma is followed by some applications. The applications can be considered as generalizations and extensions of the hyperelliptic situation.

LEMMA. *Let W be a closed Riemann surface of genus g. Suppose W admits a finite group of automorphisms, G_0, where G_0 is a group with a partition. Let the pertinent subgroups be $G_1, G_2, \cdots, G_S$. Let the order of G_i be n_i, let $W_i = W/G_i$, and let g_i be the genus of W_i for $i=0, 1, \cdots, S$. Then*

$$(1) \qquad (S-1)g + n_0 g_0 = \sum_{i=1}^{S} n_i g_i.$$

PROOF. If r_i is the ramification of the n_i-sheeted covering $W \to W/G_i$ then the Riemann-Hurwitz formula for this covering is

$$(2) \qquad 2g - 2 = n_i(2g_i - 2) + r_i.$$

Let p $(\in W)$ be a branch point in the covering $W \to W/G_0$. The elements of G_0 which leave p fixed form a cyclic subgroup of G_0 generated by, say, T. $\langle T \rangle$ lies in the G_i $(i>0)$ containing T (call it G_{i_0}) and no

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[2] The notation $\langle S, T, \cdots \rangle$ will denote the group generated by $S, T, \cdots$.

[3] For a discussion of groups with partitions see Baer [1] and Suzuki [3].

subgroup of $\langle T\rangle$ lies in any other G_i $(i>0)$. Thus the contribution of p to r_0 occurs in r_{i_0} and in no other r_i $(i>0)$. It follows that

$$r_0 = \sum_{i=1}^{S} r_i. \tag{3}$$

Moreover, a simple counting of elements shows that

$$n_0 = \sum_{i=1}^{S} n_i + 1 - S. \tag{4}$$

Formula (1) now follows by summing formula (2) as i runs from 1 to S and comparing the result to formula (2) with $i=0$ in the light of formulas (3) and (4). Q.E.D.

Formula (1) is of interest because the ramifications of the various coverings do not occur.

APPLICATION 1. Suppose W is a Riemann surface of genus three which is a smooth two-sheeted covering of a surface W_0 of genus two. Then W is hyperelliptic.[4]

PROOF. W admits an automorphism, T, of order two so that $W/\langle T\rangle = W_0$. Since W_0 is hyperelliptic, W_0 admits an automorphism, S_0, of period two so that the genus of $W_0/\langle S_0\rangle$ is zero. The composition of the two maps $W\to W_0$ and $W_0\to W_0/\langle S_0\rangle$ represents W as a four-sheeted covering of the Riemann sphere. Since $W\to W_0$ is without ramification, the branch points of $W\to W_0/\langle S_0\rangle$ are all of order one and occur in pairs, one above the other. Since $W_0/\langle S_0\rangle$ is simply-connected it follows that the covering $W\to W_0/\langle S_0\rangle$ arises from the action of a group of automorphisms, G_0, on W where G_0 is isomorphic to the noncyclic group of order four. One element of G_0 is T. G_0 admits a partition into three subgroups of order two and so formula (1) applied to this situation gives

$$g + 2g_0 = g_1 + g_2 + g_3. \tag{5}$$

Here $g_0=0$ and one g_i must be two. Consequently another g_i must be zero and so W is hyperelliptic.

APPLICATION 2. If W is a surface of genus three admitting an automorphism group, G_0, isomorphic to $Z_2\times Z_2\times Z_2$ then W is hyperelliptic.

PROOF. G_0 admits a partition with seven subgroups of order two. Formula (1) here yields

$$3g + 4g_0 = \sum_{i=1}^{7} g_i. \tag{6}$$

[4] This fact was proved by H. M. Farkas [2] using theta functions.

Since $g_0 \geqq 0$ and $g=3$ one of the g_i's must be two and so W is hyperelliptic by application one.

APPLICATION 3. If W is a surface of genus five which is a smooth two-sheeted covering of a hyperelliptic surface of genus three, then W is hyperelliptic or W can be represented as a two-sheeted covering of a torus. The proof is exactly as in application one.

The proofs of the last applications require an examination of formula (1) applied to dihedral groups. Let G_0 be a dihedral group of order $2n$. Let R generate the cyclic subgroup of order n. If V is an element of order two not in R, then $V_t = R^t V$, $(t=1, 2, \cdots, n)$ are the elements of G_0 not in $\langle R\rangle$. If G_0 is realized as a group of automorphisms on a Riemann surface W, let $W_R = W/\langle R\rangle$, $W_t = W/\langle V_t\rangle$, $t=1, 2, \cdots, n$, $W_0 = W/G_0$ with genera $g_R, g_1, \cdots, g_n, g_0$ respectively.

Formula (1) applied gives

$$ng + 2ng_0 = ng_R + 2\sum_{t=1}^{n} g_t.$$

If n is odd, then all the groups $\langle V_t\rangle$ are conjugate. Thus the W_t's are all conformally equivalent and so $g_t = g_1$ for all $t=1, 2, \cdots, n$. Thus we get

$$g + 2g_0 = g_R + 2g_1.$$

If n is even, then $\langle V_t\rangle$ is conjugate to $\langle V_s\rangle$ if and only if $t \equiv s \pmod 2$. Let g_1 and g_2 be the two genera in this case. Then,

$$g + 2g_0 = g_R + g_1 + g_2. \tag{7}$$

Actually formula (7) holds for n, odd or even, with n odd implying that $g_1 = g_2$. The similarity between formulas (7) and (5) accounts for the following applications. Of course, for $n=2$, formulas (5) and (7) are the same.

APPLICATION 4. Let g be a nonnegative integer. Let W' be a Riemann surface of genus g' so that $g' > 4g+1$. Suppose S is an automorphism of W' of order two so that the genus of $W'/\langle S\rangle$ is g. Then these properties define S uniquely and $\langle S\rangle$ is central in the full group of automorphisms of W'.

REMARK. This is a generalization of the hyperelliptic situation, $g=0$.

PROOF. Suppose S_1 and S_2 are two distinct automorphisms of W' with the properties of S. Then, S_1 and S_2 generate a dihedral group, G_0, and we may assume $S_1 = V_1$ and $S_2 = V_2$. Formula (7) gives

$$g' + 2g_0 = 2g + g_R. \tag{8}$$

The Riemann-Hurwitz formula applied to $W \to W/\langle R\rangle$ gives

$$2g' - 2 = n(2g_R - 2) + r.$$

But $n \geqq 2$ since S_1 and S_2 are distinct and $r \geqq 0$. So

$$2g' - 2 \geqq 2(2g_R - 2) \quad \text{or} \quad 2g_R \leqq g' + 1.$$

Since $g_0 \geqq 0$, we have

$$2g' + 4g_0 = 4g + 2g_R \leqq 4g + g' + 1$$

or

$$g' \leqq 4g + 1.$$

This contradiction shows that S is unique.

Let T be another automorphism of W'. Then, $T^{-1}ST$ has the same properties as S. Thus $S = T^{-1}ST$, and the proof is complete.

APPLICATION 5. Let g_1 and g_2 be nonnegative integers. Let W be a Riemann surface of genus g so that

$$2g \geqq 3g_1 + 3g_2 + 3.$$

Let W admit two distinct automorphisms S_1 and S_2, both of period two so that the genus of $W/\langle S_i\rangle$ is g_i. Then, S_1 and S_2 commute.

PROOF. Let $G_0 = \langle S_1, S_2\rangle$. Then G_0 is a dihedral group. Again set $S_1 = V_1$ and $S_2 = V_2$. Then,

$$g + 2g_0 = g_1 + g_2 + g_R.$$

Let R have order n. We wish to show that n is two, so suppose $n \geqq 3$. The Riemann-Hurwitz formula for $W \to W/\langle R\rangle$ is

$$2g - 2 = n(2g_R - 2) + r$$

or

$$2g - 2 \geqq 3(2g_R - 2)$$

or

$$3g_R \leqq g + 2.$$

Since $g_0 \geqq 0$ we have

$$3g \leqq 3g + 6g_0 = 3g_1 + 3g_2 + 3g_R$$

or

$$3g \leqq 3g_1 + 3g_2 + 2 + g.$$

This contradicts the hypothesis and thus n is two.

APPLICATION 6. Let g be a positive integer. Let W' be a Riemann surface of genus g' so that

$$g' > 3g + 2.$$

Suppose W' admits two distinct automorphisms S_1 and S_2, both with period two, so that the genus of $W'/\langle S_i\rangle$ is g, $i=1, 2$. Then S_1 and S_2 commute. Moreover, $\langle S_1, S_2\rangle$ and $\langle S_1S_2\rangle$ are normal in the full group of automorphisms of W'.[5]

PROOF. That S_1 and S_2 commute follows immediately from the previous application. To show that $\langle S_1, S_2\rangle$ and $\langle S_1S_2\rangle$ are normal, we show that there is no further S_3 with the properties of S_1 and S_2. Thus conjugation will permute S_1 and S_2 and consequently leave S_1S_2 fixed.

Therefore, suppose that S_3 is a third distinct automorphism with the properties of S_1 and S_2. Consider $G_0=\langle S_1, S_2, S_3\rangle$. If $S_3=S_1S_2$ then G_0 is isomorphic to $Z_2\times Z_2$ and formula (5) leads to the contradiction

$$g' + 2g_0 = 3g.$$

Thus G_0 is isomorphic to $Z_2\times Z_2\times Z_2$. Let the seven subgroups of order two be denoted G_i, $i=1, 2, \cdots, 7$, where $G_i=\langle S_i\rangle$ for $i=1, 2, 3$. Let g_i be the genus of W/G_i. Formula (1) becomes

$$3g' + 4g_0 = \sum_{i=1}^{7} g_i. \tag{9}$$

Now $g_1=g_2=g_3=g$. Also, as in Application 4, $2g_i\leqq g'+1$ for $i=4, 5, 6, 7$. Multiplying equation (9) by two and putting in this information we get

$$6g' \leqq 6g' + 8g_0 \leqq 6g + 4g' + 4.$$

This contradiction completes the proof.

REMARKS. We can conclude that if $g'>3g+2$ and W' admits an automorphism, S_1, of period two so that the genus of $W'/\langle S_1\rangle$ is g, then S_1 is central or else there is one other automorphism, S_2, with the properties of S_1 and S_1S_2 is central. If $g'>4g+1$ Application 4 assures us that the first alternative holds. Simple examples show that these bounds are sharp. In fact, there is a surface, W, of genus five

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[5] The author is indebted to W. T. Kiley for showing how the original form of Application 6 could be considerably strengthened to its present form.

admitting a group of automorphisms isomorphic to $Z_2 \times Z_2 \times Z_2 \times Z_2$ with five subgroups, G_i ,of order two so that the genus of W/G_i is one.

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