

A MULTILINEAR VERSION OF THE MARCINKIEWICZ INTERPOLATION THEOREM

ROBERT S. STRICHARTZ¹

It is natural to ask if there exists a multilinear version of the Marcinkiewicz interpolation theorem [3] corresponding to the multilinear Riesz interpolation theorem. If a bilinear transformation $T(f, g)$ is defined from

$$\{L^{p_0}(X, \mu) \times L^{q_0}(Y, \nu)\} + \{L^{p_1}(X, \mu) \times L^{q_1}(Y, \nu)\}$$

to measurable functions on (Z, σ) and satisfies the weak-type estimates $(p_i, q_i; r_i)$:

$$\sigma\{z: |T(f, g)(z)| \geq \alpha\} \leq M_i \left(\frac{\|f\|_{p_i} \|g\|_{q_i}}{\alpha} \right)^{r_i}$$

for $i=0, 1$, does it follow that it satisfies the strong-type estimate $(p, q; r): \|T(f, g)\|_r \leq K M_1^{1-t} M_2^t \|f\|_p \|g\|_q$ where

$$\frac{1}{p} = \frac{1-t}{p_0} + \frac{t}{p_1}, \quad \frac{1}{q} = \frac{1-t}{q_0} + \frac{t}{q_1}, \quad \frac{1}{r} = \frac{1-t}{r_0} + \frac{t}{r_1}$$

if $p \leq r, q \leq r$ and $0 < t < 1$? The answer is no, as shown by the following counterexample:

Let $X = Y = Z = (0, \infty)$ and $\mu = \nu = \sigma =$ Lebesgue measure. Let $T(f, g)(z) = \int_0^\infty f(xz)g(x)dx$. Then

$$|T(f, g)(z)| \leq \|f(xz)\|_p \|g(x)\|_q = z^{-1/p} \|f\|_p \|g\|_q$$

where $1/p + 1/q = 1$ hence we have the weak-type $(p, q; p)$ estimate holding for all $p, 1 \leq p \leq \infty$. But we never have the strong-type estimate (unless $p = \infty$). For if we choose g to be a positive function and consider the linear operator

$$S(f)(z) = \int_0^\infty f(xz)g(x)dx = \int_0^\infty f(x)g\left(\frac{x}{z}\right)\frac{dx}{z},$$

it is an integral operator with positive kernel homogeneous of degree -1 , hence by Hardy-Littlewood-Pólya [2, Chapter 9] will be bounded in L^p if and only if $\int_0^\infty g(x)x^{-1/p}dx < \infty$ which is not true for every $g \in L^q$.

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If we are willing to strengthen the hypotheses, however, by adding a third point, we can obtain the desired conclusion as follows:

THEOREM. *Let $T(f, \dots, g)$ be an m -linear transformation from $\{L^{p_0}(X, \mu) \times \dots \times L^{q_0}(Y, \nu)\} + \dots + \{L^{p_m}(X, \mu) \times \dots \times L^{q_m}(Y, \nu)\}$ to measurable function on (Z, σ) . Suppose T satisfies the $m+1$ weak-type $(p_i, \dots, q_i; r_i)$ estimates for $i=0, \dots, m$. Suppose the $m+1$ points $(1/p_i, \dots, 1/q_i)$ in m -space span a nondegenerate simplex, and let $(1/p, \dots, 1/q)$ be a point in the interior of the simplex. In barycentric coordinates*

$$\frac{1}{p} = \sum \frac{s_i}{p_i}, \dots, \frac{1}{q} = \sum \frac{s_i}{q_i}$$

where $\sum s_i = 1$ and $0 < s_i < 1$. Let $1/r = \sum s_i/r_i$. Suppose that for just one of the p, \dots, q variables, say p , we have $p \leq r$ and that r is non-constant as p varies with the other variables fixed. Then T satisfies the strong-type $(p, \dots, q; r)$ estimate.

For simplicity we will consider only the case of bilinear transformations. The proof in the general case is entirely analogous. We need the following lemma, which follows from the work of Calderón [1, Theorems 10.1 and 13.6] in case $r_0, r_1 > 1$. We shall give a proof suggested by E. M. Stein. In what follows we use the properties of Banach space valued analytic functions [4] and Lorentz spaces [5].

LEMMA. *Let T be a bilinear transformation defined on simple functions satisfying a weak-type $(p_i, q_i; r_i)$ estimate with constant M_i for $i=0, 1$, where $p_i, q_i, r_i \geq 1$ and $r_0 \neq r_1$. Then T satisfies a weak-type $(p, q; r)$ estimate with constant $M_0^{1-t} M_1^t$.*

PROOF. If f is any measurable function on (Z, σ) such that

$$\sigma\{z: |f(z)| > \alpha\}$$

is finite for all $\alpha > 0$, let $f^*(t) = \sup\{\alpha: \sigma\{z: |f(z)| > \alpha\} \geq t\}$ be the nonincreasing rearrangement of f . Let $\|f\|_{r,\infty}^* = \sup_{t>0} t^{1/r} f^*(t)$ and $\|f\|_{r,1}^* = \int_0^\infty t^{1/r-1} f^*(t) dt$, and denote by $L^{r,\infty}$ and $L^{r,1}$ the spaces of functions with $\|f\|_{r,\infty}^*$ and $\|f\|_{r,1}^*$ finite. Although these are not norms, they are equivalent to norms if $r > 1$, and $L^{r,\infty}$ is the dual space of $L^{r',1}$, where $1/r + 1/r' = 1$ (note $L^{\infty,\infty} = L^\infty$ and $L^{1,1} = L^1$). Furthermore

$$\|f\|_{r,\infty}^* = \sup \left\{ \left| \int fg d\sigma \right| : \|g\|_{r',1}^* \leq 1 \right\} [5].$$

It is easy to see that a weak-type $(p, q; r)$ estimate is equivalent to $\|T(f, g)\|_{r, \infty}^* \leq M \|f\|_p \|g\|_q$. We note that

$$\|T(f, g)\|_{r, \infty}^* = (\| |T(f, g)|^{1/2} \|_{2r, \infty}^*)^2 \quad \text{and } 2r > 1.$$

Thus to prove the lemma we must show

$$\int |T(f, g)|^{1/2} h d\sigma \leq (M_0^{1-t} M_1^t)^{1/2}$$

for $\|f\|_p \leq 1$, $\|g\|_q \leq 1$ and $\|h\|_{u, 1} \leq 1$ where $1/u + 1/2r = 1$, given that the same inequality holds for $t=0$ and $t=1$.

The proof now proceeds as the proof of the Riesz-Thorin theorem, except that some modifications are necessary since $|T(f, g)|^{1/2}$ is not linear. We may assume (Z, σ) is nonatomic, since otherwise we may replace each atom Z_0 by an interval of the same measure and set $T(f, g) = T(f, g)(Z_0)$ on that interval. We let

$$f_s = |f|^{p((1-s)/p_0 + s/p_1)} \arg f \quad \text{and} \quad g_s = |g|^{q((1-s)/q_0 + s/q_1)} \arg g.$$

We choose a function ϕ on (Z, σ) equimeasurable with $1/t$ on $((0, \sigma(Z)), dt)$ such that $(\phi^\alpha h)^* = (1/t^\alpha) h^*(t)$ for every $\alpha > 0$. Then let

$$H_s = \phi^{(1-s)/u_0 + s/u_1 - 1/u} \cdot h.$$

Finally let

$$F(s) = |M_0^{1-s} M_1^s|^{-1/2} \int |T(f_s, g_s)|^{1/2} |H_s| d\sigma.$$

It is easy to show that our hypotheses on T imply $|F(i\tau)| \leq 1$ and $|F(1+i\tau)| \leq 1$ for all real τ , and the desired conclusion is $|F(t)| \leq 1$. This will follow from the Phragmén-Lindelöf principle if we can show that F is bounded and subharmonic in the strip $0 \leq \text{Re}(S) \leq 1$.

Boundedness is immediate, since $|T(f_s, g_s)|^{1/2}$ is bounded in $L^{2r_0, \infty} \cap L^{2r_1, \infty}$ and H_s is bounded in $L^{u_0, 1} + L^{u_1, 1}$. Next we claim that for almost every $z \in Z$, the function $T(f_s, g_s)(z)$ is analytic in s (after redefining these functions on sets of measure zero). Indeed $T(f_s, g_s)$ is a holomorphic Banach-space valued function (since either r_0 or r_1 is greater than 1) hence Cauchy's integral formula

$$T(f_s, g_s) = \frac{1}{2\pi i} \int_C \frac{T(f_t, g_t)(z)}{s-t} dt$$

is valid. But

$$\frac{1}{2\pi i} \int_C \frac{T(f_t, g_t)(z)}{s-t} dt$$

is holomorphic in s for fixed z provided $\int_C |T(f_t, g_t)(z)| dt < \infty$, which holds almost everywhere since $\int_C \int_B |T(f_t, g_t)(z)| dz dt < \infty$ for every $B \subset Z$ of finite measure.

Thus $F(s)$ is an integral of square roots of moduli of holomorphic functions, hence is subharmonic.

PROOF OF THEOREM. We apply the lemma to conclude that T satisfies a weak-type $(p, q; r)$ estimate for every point in the interior of the triangle. We then consider two points $(p + \epsilon, q; r_1)$ and $(p - \epsilon, q; r_2)$ in the triangle and apply the ordinary Marcinkiewicz interpolation theorem to $S(f) = T(f, g)$ to obtain the strong $(p, q; r)$ estimate, noting that we have $p \leq r$ and $r_1 \neq r_2$ by hypothesis.

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY