# SMOOTHING IN $C(X)$

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1. In this note we prove convergence theorems for smoothing operators (conditional expectations) in the space of continuous functions. In this setting conditions which imply uniform convergence are found; indeed, in the form that the problem is set the conditions are necessary and sufficient. We give examples showing that many of the properties of conditional expectations on a measure space no longer hold in $C(X)$.

2. The space of all continuous real valued functions on a compact Hausdorff space $X$ is, as usual, $C(X)$. An operator $T$ on $C(X)$ is a *Markov operator* if $\|T\| = T1 = 1$. The following is well known [1]; if the range of a Markov projection is a subalgebra of $C(X)$ the smoothing equation $T(fTg) = (Tf)(Tg)$ is satisfied and conversely if a Markov projection satisfies the smoothing equation then the range is an algebra. A subalgebra $A$ of $C(X)$ will be called *smooth* if there is a Markov projection onto $A$. The terminology is somewhat misleading; smoothness depends on the embedding of $A$ in $C(X)$ rather than intrinsic properties of $A$. Necessary and sufficient conditions for a subalgebra to be smooth are not known. However see Michael [3] for a sufficient condition and see Lloyd [2] for various examples. If we consider the map to constant functions defined by $f \rightarrow (\mu, f)$ where $\mu$ is any probability on $X$ we see that a smooth subalgebra may fail to have a unique smoothing projection.

Let $X$ consist of $[0, 1] \cup \{1+1/n : n \in N\} \cup \{-1/n : n \in N\}$. Let $A_1$ be the $C(X)$ subalgebra of functions constant on $\{1+1/n : n \in N\} \cup \{-1\}$ and $A_2$ the functions constant on $\{-1/n : n \in N\} \cup \{2\}$. Then $A_1$ and $A_2$ are both smooth subalgebras (and here we do have uniqueness of the smoothing operations) and $A_1 \cap A_2$ fails to be smooth.

For the convergence theorems we will consider a nested sequence of smooth subalgebras, either increasing or decreasing. For the increasing sequence we define the *final algebra* $A_\infty = \bigcup A_m^-$ and in the decreasing case the final algebra is $A_\infty = \bigcap A_n$. Neither is in general smooth. For with $X$ as above we let $A_n$ be the subalgebra of $C(X)$ consisting of functions constant on the subset of $X$ contained in $[-1, 2] \setminus (-1/n, 1+1/n)$. For an increasing nest we take $A_n$ to be

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constant on $[-1/n, 1+1/n]\cap X$. In both cases the final algebras are not smooth.

3. If $P$ is a Markov projection there is a non-empty compact set $K$ of probabilities invariant under $T^*$ (Tychonov). Since uniqueness of projections in general fails even when existence is known we will assume that the projections commute.

LEMMA. *Let $A_1 \subset A_2$ be two smooth subalgebras with given projections $P_1$ and $P_2$. Then $P_1$ and $P_2$ commute iff $K_1 \subset K_2$.*

**Proof.** We suppose first $P_1P_2 = P_2P_1$. Let $\mu$ be in $K_1$. Then $(f, P_2^*\mu) = (f, P_2^*P_1^*\mu)(f, P_1^*P_2^*\mu) = (P_1f, P_2^*\mu) = (P_2P_1f, \mu) = (P_1f, \mu)(f, P_1^*\mu) = (f, \mu)$. So we have $P_2\mu = \mu$. Now conversely suppose $K_1 \subset K_2$. Then $P_1P_2f(x) = (P_1P_2f, \mathcal{E}_x) = (P_2P_1f, P_1^*\mathcal{E}_x) = (f, P_2^*P_1^*\mathcal{E}_x) = (f, P_1^*\mathcal{E}_x) = (P_1f, \mathcal{E}_x) = P_1f(x) = P_2P_1f(x)$.

Now if the projections commute the invariant sets $K_n$ are nested so we define $K_\infty = \bigcap K_n$ in the decreasing case and $K_\infty = \bigcup K_n^-$ in the increasing case.

THEOREM. *Let $\{P_n\}$ be a commutative set of smoothing projections associated with a nested sequence of smooth subalgebras. Then $P_n$ converges in the strong operator topology iff $A_\infty$ separates the points of $K_\infty$. The operator defined by the limit is then a Markov smoothing projection onto $A_\infty$.*

**Proof.** First we consider $A_n \downarrow A_\infty$. Let $\mathcal{E}(x)$ be the unit point mass at $x$ and let $\mu(x)$ be a $\omega^*$ cluster point of $\{P_n^*\mathcal{E}(x)\}$; since $P_n^*\mathcal{E}(x)$ is in $K_n$ we have $\mu(x)$ in $K_\infty$. Now if $f$ is in $A_\infty$ then $P_nf(x)$ is constant. So since $A_\infty$ separates the points of $K_\infty$, $P_n^*\mathcal{E}(x)$ has a unique cluster point so $P_n^*\mathcal{E}(x) \to \mu(x)$. The map $x \to \mu(x)$ is continuous in the topology of $K_\infty$ defined by the $A_\infty$ functions. However, this is just the $\omega^*$ topology restricted to $K_\infty$ (the $A_\infty$ topology is a weaker Hausdorf topology for the compact space $K_\infty$). By virtue of the $\omega^*$ continuity of $x \to \mu(x)$ we can define the (necessarily) Markov operator $P_\infty f(x) = (f, \mu(x))$. We have $P_n \to P_\infty$ in the weak operator topology and wish to show uniform convergence of $P_nf$. Suppose $P_\infty g = 0$ so $P_ng$ goes to zero pointwise. If $\|P_ng\| = r(n) > \delta > 0$ we can always pick a probability $\mu(n)$ in $K_n$ so that $|(P_ng, \mu(n))| = r(n)$. For this note that $P_ng$ is invariant under $P_n$. Pick $x$ so $|P_ng(x)| = r(n)$ and let $\mu(n)$ be the Cesaro limit of $\mathcal{E}(x)$. Now we can pass to a subnet so that $P_\alpha\mu(\alpha) \to \mu(0)$ in $K_\infty$. But $|(g, P_\alpha^*\mu(\alpha))| \to |(g, \mu(0))| = |(g, P_\infty^*\mu(0))| = 0$. Contradiction.

The increasing case is similar. We have $P_n^*\mathcal{E}(x)$ in $K_n$ so separation gives a unique cluster point and so convergence. Again continuity of

$x \to \mu_x$ yields a weak operator limit $P_\infty$. Now if $P_\infty g = 0$ and $\|P_n g\| = r(n) > \delta > 0$ we again pick a $P_n$-invariant probability $\mu(n)$ with $|(\mu(n), P_n g)| = r(n)$. Passage to a convergent subset again gives the contradiction.

If uniform convergence of $P_n f$ holds then we obtain a Markov idempotent with range an algebra $A_\infty$. If $\mu_1$ and $\mu_2$ are two $P_\infty$-invariant probabilities we pick $f$ to be any $C(X)$ function distinguishing $\mu_1$ and $\mu_2$. The function $P_\infty f$ is in $A_\infty$ and clearly still distinguishes $\mu_1$ and $\mu_2$.

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