AUTOMORPHISMS OF THE FUNDAMENTAL GROUP OF A CLOSED, ORIENTABLE 2-MANIFOLD[1]

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In the following pages we note a relationship which exists between the mapping class group (or homeotopy group) of a closed, orientable 2-manifold and Artin's braid group. Let T_g be a closed, orientable 2-manifold of genus g, and let $\pi_1 T_g$ be its fundamental group. Let $M(T_g)$ be the mapping class group of T_g, that is the group of all isotopy classes in the space of all orientation-preserving homeomorphisms of $T_g \to T_g$, or equivalently [4] the group Aut $\pi_1 T_g/\text{Inn } \pi_1 T_g$. Let B_n be the Artin braid group, that is the abstract group on $n-1$ generators $\sigma_1, \cdots, \sigma_{n-1}$ with defining relations

$$\sigma_i \leftrightarrows \sigma_j, \qquad i = 1, 2, \cdots, n-1, \quad |i-j| \geqq 2,$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \qquad i = 1, 2, \cdots, n-2.$$

The group $M(T_1)$ is the modular group, which is a homomorphic image of B_3 and also (with an appropriate choice of generators) of B_4. It was shown by Bergau and Mennicke [1] that $M(T_2)$ is a homomorphic image of B_6. We are able to generalize these known results to show that $M(T_g)$ contains a sequence of $(n-1)$-generator subgroups which are homomorphic images of B_n for $n=3, 4, \cdots, 2g+2$, the union of these subgroups generating all of $M(T_g)$. If $g>2$, all of these subgroups are proper in $M(T_g)$.

PROOF. Generators for $M(T_g)$ are known, and we use a particularly simple set which was discovered by M. Dehn [2] and recently rediscovered and simplified by W. B. R. Lickorish [3]. Let A be an annulus in the Euclidean plane, parametrized by (r, θ), where

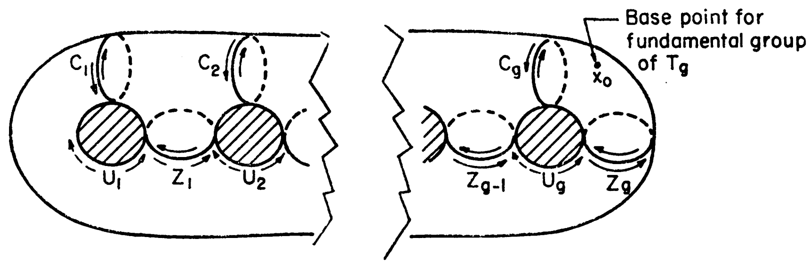

FIGURE 1

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$1 \leqq r \leqq 2$ and θ is a real number mod 2π. We define a homeomorphism $h: A \rightarrow A$ by $h(r, \theta) = (r, \theta - 2\pi r)$. If C is now any simple closed curve on T_g, and $e_C: A \rightarrow T_g$ an imbedding of $A \rightarrow$ a neighborhood of C in T_g, which maps $r = 1.5$ onto C, then $e_C h e_C^{-1} / e_C A$ may be extended by the identity map on $(T_g - e_C A)$ to a homeomorphism $h_C: T_g \rightarrow T_g$, which is called a twist about C. It was shown in [3] that the twists $\{h_{U_i}, h_{Z_i}, h_{C_j}; 1 \leqq i \leqq g, 1 \leqq j \leqq g-1\}$ about the curves $\{U_i, Z_i, C_j\}$ in Figure 1 generate $M(T_g)$. The arrows indicate the direction of relative motion of points on T_g when each such twist is performed.

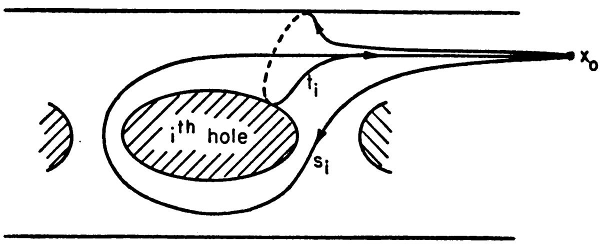


FIGURE 2

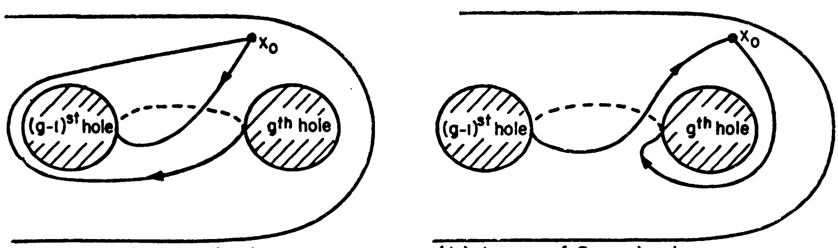
Each of these twists induces an automorphism of $\pi_1 T_g$, and these can be determined explicitly as follows. We take a base point for $\pi_1 T_g$ as indicated in Figure 1, and define generators $\{t_i, s_i; i = 1, 2, \cdots, g\}$ for $\pi_1(T_g, x_0)$ which are represented by the loops illustrated in Figure 2. In terms of these generators the group Aut $\pi_1 T_g$ has the single defining relation

$$\prod_{i=1}^{g} s_i^{-1} \tilde{t}_i s_i \tilde{t}_i^{-1} = 1$$

where

$$\tilde{t}_1 = s_1 t_1 s_1^{-1}, \quad \tilde{t}_i = s_i(\tilde{t}_{i-1} t_{i-1}^{-1} t_i) s_i^{-1} \quad \text{if } i \neq 1.$$

The automorphism induced by a typical twist $h_{Z_{g-1}}$ on the t_i, s_i will now be determined. Clearly $h_{Z_{g-1}}$ will not effect any of the t_i, nor will



(a) Image of S_{g-1} under $h_{Z_{g-1}}$ (b) Image of S_g under $h_{Z_{g-1}}$

FIGURE 3

$h_{Z_{g-1}}$ effect s_i if $i \neq g$ or $g-1$. To determine the effect of $h_{Z_{g-1}}$ on s_{g-1} and s_g, we observe that, under the action of $h_{Z_{g-1}}$: s_{g-1} and s_g will be altered to the curves illustrated in Figures 3a and 3b respectively, which can be seen to be homotopic to the curves $t_{g-1}^{-1}t_g s_{g-1}$ and $s_g t_g^{-1} t_{g-1}$ on T_g. The induced automorphisms for all possible cases can be determined from similar pictures, and are found to be:

$$
\begin{aligned}
h_{U_i}&: t_i \to t_i s_i \\
h_{C_i}&: s_j \to t_i s_j t_i^{-1} && \text{if } j < i \\
&\quad t_j \to t_i t_j t_i^{-1} && \text{if } j < i \\
&\quad s_i \to s_i t_i^{-1} \\
h_{Z_i}&: s_i \to t_i^{-1} t_{i+1} s_i && \text{if } i = 1, 2, \cdots, g-1 \\
&\quad s_{i+1} \to s_{i+1} t_{i+1}^{-1} t_i && \text{if } i = 1, 2, \cdots, g-1 \\
h_{Z_g}&: s_g \to t_g^{-1} s_g
\end{aligned}
$$

where it is understood that every generator which is not listed explicitly is unaltered by the h's.

Since $\{h_{U_i}, h_{Z_i}, h_{C_j};\ 1 \leqq i \leqq g,\ 1 \leqq j \leqq g-1\}$ generate $M(T_g)$, the automorphisms listed above generate Aut $\pi_1 T_g/\text{Inn } \pi_1 T_g$. This provides a very simple tool for verifying relations in $M(T_g)$: one simply calculates the automorphism induced by a word one suspects is a relator, and observes whether this automorphism is an inner automorphism of $\pi_1 T_g$. Using this procedure, we find the following relations hold among the $h_{U_i}, h_{Z_i}, h_{C_j}$.

(1) $h_{U_i} \leftrightarrows h_{U_j}, \qquad i \neq j,$

(2) $h_{C_i} \leftrightarrows h_{C_j}, \qquad i \neq j,$

(3) $h_{Z_i} \leftrightarrows h_{Z_j}, \qquad i \neq j,$

(4) $h_{U_i} \leftrightarrows h_{C_j}, \qquad i \neq j,$

(5) $h_{Z_i} \leftrightarrows h_{C_j}, \qquad i, j = 1, 2, \cdots, g,$

(6) $h_{U_i} \leftrightarrows h_{Z_j}, \qquad i \neq j-1 \text{ or } j,$

(7) $h_{U_i} h_{Z_i} h_{U_i} = h_{Z_i} h_{U_i} h_{Z_i}, \qquad i = 1, 2, \cdots, g,$

(8) $h_{U_{i+1}} h_{Z_i} h_{U_{i+1}} = h_{Z_i} h_{U_{i+1}} h_{Z_i}, \qquad i = 1, 2, \cdots, g-1,$

(9) $h_{C_i} h_{U_i} h_{C_i} = h_{U_i} h_{C_i} h_{U_i}, \qquad i = 1, 2, \cdots, g,$

(10) $(h_{C_1} h_{U_1} h_{Z_1} h_{U_2} h_{Z_2} \cdots h_{U_g} h_{Z_g})^{2g+2} = 1,$

(11) $(h_{C_1} h_{U_1} h_{Z_1} h_{U_2} h_{Z_2} \cdots h_{U_g} h^2_{Z_g} h_{U_g} \cdots h_{Z_2} h_{U_2} h_{Z_1} h_{U_1} h_{C_1})^2 = 1.$

We cannot say whether this system is complete.

It then follows that there is a homomorphism ϕ: $B_{2g+2} \to M(T_g)$ defined by $\phi(\sigma_1) = h_{C_1}$, $\phi(\sigma_{2j}) = h_{U_j}$, $\phi(\sigma_{2j+1}) = h_{Z_j}$, for $j = 1, 2, \cdots, g$.

Moreover, since the center of B_n is known to be infinite cyclic and generated by $(\sigma_1\sigma_2 \cdots \sigma_{n-1})^n$, in view of relation (10) we have that the center of $B_{2g+2} \subset \ker \phi$. This is not, however, the full kernel because relation (11) also holds in $M(T_g)$, and from previous studies of the group B_n it is known that relation (11) is not a consequence of the others in B_n/center.

Moreover, the $(2g-2i+3)$-generator subgroups of $M(T_g)$ generated by $\{h_{C_i}, h_{U_i}, h_{Z_i}, h_{U_{i+1}}, h_{Z_{i+1}}, \cdots, h_{U_g}, h_{Z_g}\}$ are homomorphic images of $B_{2g-2i+4}$ for each $i=2, \cdots, g$ under the homomorphism $\phi_i\colon B_{2g-2i+4} \rightarrow M(T_g)$ defined by $\phi_i(\sigma_1) = h_{C_i}$, $\phi_i(\sigma_{2j}) = h_{U_{j+i-1}}$, $\phi_i(\sigma_{2j+1}) = h_{Z_{j+i-1}}$ for $j=1, 2, \cdots, g-i+1$.

Finally, the $(2g-2i+2)$-generator subgroups of $M(T_g)$ which are generated by $\{h_{U_i}, h_{Z_i}, h_{U_{i+1}}, h_{Z_{i+1}}, \cdots, h_{Z_g}\}$ are homomorphic images of $B_{2g-2i+3}$ for each $i=1, 2, \cdots, g$ under the homomorphism ψ_i defined by $\psi_i(\sigma_{2j-1}) = h_{U_{j+i-1}}$, $\psi_i(\sigma_{2j}) = h_{Z_{j+i-1}}$ for $j=1, 2, \cdots, g-i+1$.

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