COMPLETENESS PRESERVING MULTIPLIERS[1]

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Given a $\phi(x) \in L^\infty(-\pi, \pi)$, so that ϕ is a multiplier on $L^2(-\pi, \pi)$, it is interesting to ask when the following implication holds:

(1) If $\{\psi_n\}_{n=-\infty}^{\infty}$ is any complete orthonormal set (CONS) for L^2 and if S is any subset of the integers, then the new set $\{\varphi_n\}_{-\infty}^{\infty}$ defined by $\varphi_n = \psi_n$ for $n \in S$, $\varphi_n = \phi \cdot \psi_n$ for $n \notin S$, is also complete in L^2.

The following theorem gives a rather simple necessary and sufficient condition for (1).

THEOREM I. (1) *holds if and only if there exists a complex number* α *such that*

$$(*) \qquad \begin{array}{l} \text{(i) } \operatorname{Re} \alpha\phi \geqq 0 \textit{ almost everywhere (a.e.), and} \\ \text{(ii) } \textit{either } \operatorname{Im} \alpha\phi > 0 \textit{ a.e. or } \operatorname{Im} \alpha\phi < 0 \textit{ a.e.} \\ \qquad \textit{on the zero set } \boldsymbol{Z} \textit{ of } \operatorname{Re} \alpha\phi. \end{array}$$

PROOF. Suppose first that (*) holds. Let f be any L^2 function orthogonal to all the φ_n, so that

$$\int_{-\pi}^{\pi} f(x)\psi_n(x)^- dx = 0 \quad \text{for } n \in S \quad \text{and} \tag{2}$$

$$\int_{-\pi}^{\pi} f(x)\bar{\alpha}\phi(x)^- \psi_n(x)^- dx = 0 \quad \text{for } n \in T \tag{3}$$

(where $g(x)^-$ denotes the complex conjugate of $g(x)$ and $T = \text{comp}(S)$).

Now (2) says that the Fourier series of f is given by

$$f(x) \sim \sum_{n \in T} a_n \psi_n(x),$$

and if we let the partial sum of this Fourier series be

$$S_N(f, x) = \sum_{n \in T;\, |n| \leq N} a_n \psi_n(x) \tag{4}$$

we see that (3) and (4) yield

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$$\int_{-\pi}^{\pi} f(x)\bar{\alpha}\phi(x)^{-}S_N(f, x)^{-}dx = 0, \qquad N = 1, 2, 3, \cdots. \tag{5}$$

By the L^2 convergence of the Fourier series we can let $N \to \infty$ in (5), and we get

$$\int_{-\pi}^{\pi} |f(x)|^2\bar{\alpha}\phi(x)^{-}dx = 0. \tag{6}$$

Thus $f=0$ a.e. where Re $\alpha\phi>0$, so that

$$\int_Z |f(x)|^2\bar{\alpha}\phi(x)^{-}dx = \int_Z |f(x)|^2 \operatorname{Im} \bar{\alpha}\phi(x)^{-}dx = 0. \tag{7}$$

Combining the above with (ii) we see that $f=0$ a.e., so $\{\varphi_n\}$ is complete.

We now assume that (*) is false and thereby produce a function $\omega(x) \in L^1(-\pi, \pi)$ satisfying

$$\omega(x) \geqq 0, \qquad \int_{-\pi}^{\pi} \omega(x)dx = 1, \quad \text{and} \quad \int_{-\pi}^{\pi} \omega(x)\phi(x)dx = 0. \tag{8}$$

Indeed, suppose no such $\omega(x)$ exists. Then the linear functional $\Lambda: f \to \int_{-\pi}^{\pi} f(x)\phi(x)dx$ on $L^1(-\pi, \pi)$ is never 0 on the convex set $K=\{\omega \in L^1: \omega \geqq 0, \int_{-\pi}^{\pi} \omega(x)dx=1\}$, so that $\Lambda(K)$ is a convex subset of the plane which misses 0. This assures the existence of a complex number α satisfying

$$\operatorname{Re} \alpha \int_{-\pi}^{\pi} \omega(x)\phi(x)dx \geqq 0 \qquad \text{for every } \omega \in K \quad \text{and} \tag{9}$$

$$\alpha \int_{-\pi}^{\pi} \omega(x)\phi(x)dx \neq 0 \qquad \text{for any } \omega \in K. \tag{10}$$

By (9) we see that (i) holds. But (ii) must also hold, since all other cases are clearly excluded by (10). Thus (*) holds, and this contradiction establishes the existence of a function $\omega \in L^1(-\pi, \pi)$ satisfying (8).

If we now let $\{\psi_n\}$ be any CONS with $\psi_0 = (\omega(x))^{1/2}$ and choose $T=\{0\}$ then the set $\{\varphi_n\}$ which is thereby generated is orthogonal to the function $(\omega(x))^{1/2}$. This means indeed that (1) does not hold, and completes the proof.

If we no longer consider an arbitrary CONS but restrict our attention to the standard one $\psi_n(x) = e^{inx}$ our multiplier can be any L^2 function, it need not be bounded. Thus, for each $\phi \in L^2$, we wish to know if the following implication holds:

(11) If we break up the integers arbitrarily into two disjoint sets S and T, and if we let $\varphi_n(x)=e^{inx}$ for $n\in S$ and $\varphi_n(x)=\phi(x)e^{inx}$ for $n\in T$, then $\{\varphi_n\}$ is complete in L^2.

In this situation the question of necessary and sufficient conditions is left open. It appears to be a difficult one.

If $\phi(x)$ is bounded and satisfies (*) for some α we see that Theorem I applies and (11) holds. If we assume only that (*) holds without requiring that ϕ be bounded then (11) need not hold, as the following example demonstrates.

Example (i). We define two functions f and ϕ by

$$f(z)=\frac{1}{(1-z)^{1/3}} \quad \text{and} \quad \phi(z)^- = \frac{|1-z|^{2/3}}{1-z}. \tag{12}$$

Since Re $(1/(1-z))=\frac{1}{2}$ for $|z|=1$ we see that

$$\operatorname{Re}\phi(z)>0 \text{ a.e.} \qquad \text{for } |z|=1. \tag{13}$$

In addition, it is obvious from (12) that

$$f\in L^2(-\pi,\pi) \quad \text{and} \quad \phi\in L^2(-\pi,\pi). \tag{14}$$

Furthermore, for $z=e^{ix}$ and $\bar z=e^{-ix}$ we have

$$\begin{aligned} f(z)\phi(z)^- &= \frac{|1-z|^{2/3}}{(1-z)^{4/3}} = \frac{(1-z)^{1/3}(1-\bar z)^{1/3}}{(1-z)^{4/3}} \\ &= \frac{(1-\bar z)^{1/3}}{1-1/\bar z} = \frac{-\bar z}{(1-\bar z)^{2/3}}. \end{aligned} \tag{15}$$

But (12) and (15) show that, when looked at as functions on the circle $-\pi\leqq x<\pi$, f has no negative Fourier coefficients and $f\cdot\bar\phi$ has no nonnegative Fourier coefficients, i.e.,

$$\begin{aligned} &\int_{-\pi}^{\pi} f(e^{ix})e^{-inx}dx = 0 \qquad \text{for } n<0 \quad \text{and} \\ &\int_{-\pi}^{\pi} f(e^{ix})\phi(e^{ix})^- e^{-inx}dx = 0 \qquad \text{for } n\geqq 0. \end{aligned} \tag{16}$$

Thus, if we take $\varphi_n(x)=e^{inx}$ for $n<0$ and $\varphi_n(x)=\phi(e^{ix})e^{inx}$ for $n\geqq 0$ we see from (13), (14) and (16) that we have a counterexample to (11).

The following theorem shows that (11) is equivalent to an interesting uniqueness property for l_2 series.

THEOREM II. (11) *is equivalent to*:

(17) *Suppose that* $\{a_n\} \in l_2$ *and* $\{c_n\} \in l_2$, *that* $\phi(x) = \sum_{n=-\infty}^{\infty} c_n e^{-inx}$ *and that* $a_n(a*c)_n \equiv 0$, *where* $(a*c)_n = \sum_{k=-\infty}^{\infty} a_k c_{n-k}$. *Then* $a_n \equiv 0$.

(*Note*. By equivalent we mean that (11) holds for $\phi(x)$ if and only if (17) holds for the same $\phi(x)$.)

The trivial proof, which consists of showing that a counterexample to (11) leads directly to a counterexample to (17) and viceversa, is omitted.

If we state Theorem I in terms of (17) we get

THEOREM III. *Let* $\{a_n\} \in l_2$ *and* $\{c_n\} \in l_2$ *and define* $\phi(x) = \sum_{n=-\infty}^{\infty} c_n e^{-inx}$. *Suppose that* ϕ *is bounded, that* (*) *holds for some* α, *and that* $a_n(a*c)_n \equiv 0$. *Then* $a_n \equiv 0$.

One special set of circumstances under which (17) holds is given by:

THEOREM IV. *Suppose that in addition to the hypotheses of* (17) *we also have* (*) *holds and* $\{(a*c)_n\} \in l_2$. *Then* $a_n \equiv 0$.

PROOF. By hypotheses $\sum_{n=-\infty}^{\infty} \bar{a}_n(a*c)_n = 0$. Since $\{(a*c)_n\} \in l_2$ we can apply Parseval's formula to this equation, and we get

$$\int_{-\pi}^{\pi} |f(x)|^2 \phi(x)dx = 0 \qquad \text{where } f(x) = \sum_{n=-\infty}^{\infty} a_n e^{-inx}. \tag{18}$$

Since (*) holds (18) shows that $f=0$ a.e., or $a_n \equiv 0$.

Note that Theorem IV shows that if we have a counterexample to (11) with (*) holding the L^2 function f which is orthogonal to all the φ_n cannot be such that $f\cdot\phi \in L^2$.

The questions with which we dealt above can also be asked for function spaces other than $L^2(-\pi, \pi)$. In particular, if we are given a $\phi \in L^p(-\pi, \pi)$ for some p, $1 \leqq p \leqq \infty$, we can ask whether $\{\varphi_n\}$ is complete in L^p, where φ_n are those given in (11). With a few minor modifications the proof of the "if" half of Theorem I applies to the following result.

THEOREM V. *Suppose that* $1 \leqq p < 2$, *that* (*) *holds for some* α, *and that* $\phi \in L^{p/(2-p)}$. *Then* $\{\varphi_n\}$ *is complete in* L^p.

If $1 < p < \infty$ and a is any number such that

$$(p-1)/2p < a < (p-1)/p = 1/q, \tag{19}$$

and if we define two functions f and ϕ by

$$f(z) = \frac{1}{(1-z)^a} \quad \text{and} \quad \phi(z)^- = \frac{|1-z|^{2a}}{1-z}, \tag{20}$$

then seen we see that, using the method of Example (i), we have a counterexample to (11) for $L^p(-\pi, \pi)$ with $1<p<\infty$, where $\operatorname{Re}\phi>0$ a.e. Note that if $p<2$ (19) and (20) show that the number $p/(2-p)$ given in Theorem V is best possible. If $p>2$ (19) assures that we can take $a>\frac{1}{2}$, so that in this case we actually do have a bounded counterexample!

Finally, we consider the space C of continuous functions on the circle (i.e., continuous and periodic with period 2π), and we ask whether $\{\varphi_n\}$ given by (11) is complete in C.

If ϕ is "smooth" enough and satisfies the usual "direction property" we see by the following theorem that we have completeness.

THEOREM VI. *Suppose ϕ is a C^2 function on $[-\pi, \pi]$ and* $\operatorname{Re}\alpha\phi>0$ *for some α. Then $\{\varphi_n\}$ is complete in C.*

To prove this theorem we observe that the Fourier coefficients c_n of ϕ certainly satisfy $\sum_{n=-\infty}^{\infty}|n|^{1/2}|c_n|<\infty$ and then apply a result given in [1]. Since it would require the development of a new topic to even state this result we refer the reader to [1, Corollary III.2].

We conclude our work by producing a counterexample to Theorem VI when we do not assume the added restriction that $\phi\in C^2$.

Example (ii). We construct a function ϕ and a nonzero measure dy such that:

$$\phi \in C \quad \text{and} \quad \operatorname{Re}\phi > 0, \tag{21}$$

$$\int_{-\pi}^{\pi} e^{-inx}dy(x) = 0 \qquad \text{for } n > 0 \quad \text{and} \tag{22}$$

$$\int_{-\pi}^{\pi} \phi(x)e^{-inx}dy(x) = 0 \qquad \text{for } n \leqq 0. \tag{23}$$

(22) is equivalent to

$$dy(x) = f(e^{ix})^{-}dx \qquad \text{where } f \in H^1 \text{ of the unit disk.} \tag{24}$$

Combining (23) and (24) we get $\phi(x)f(e^{ix})^{-}=e^{ix}g(e^{ix})$ where $g\in H^1$. Thus, we want to find two functions f and g such that

$$f \text{ and } g \text{ are in } H^1 \text{ of the disk and} \tag{25}$$

$$\phi(z) = zg(z)/\bar{f}(z) \text{ is continuous and has positive real part} \tag{26}$$

for $z=e^{ix}$, $-\pi\leqq x\leqq\pi$. This is done by letting

$$f(z) = g(z) = i(1 - z)^{-1}\log^{-3/2}\left(\frac{N}{1 - z}\right) \tag{27}$$

where N is any positive number large enough so that

$$\text{(28)} \qquad \operatorname{Re} \log^3\left(\frac{N}{1-z}\right) > 0 \qquad \text{for } z = e^{ix}, \qquad -\pi \leqq x \leqq \pi.$$

It is obvious that f and g satisfy (25), and we have

$$\text{(29)} \qquad \begin{aligned} \phi(z) &= \frac{zf}{\bar{f}} = \frac{zf^2}{|f|^2} = \frac{-z(1-z)^{-2}\log^{-3}\left(\frac{N}{1-z}\right)}{|1-z|^{-2}\left|\log\left(\frac{N}{1-z}\right)\right|^{-3}} \\ &= \frac{\log^{-3}\left(\frac{N}{1-z}\right)}{\left|\log\left(\frac{N}{1-z}\right)\right|^{-3}}. \end{aligned}$$

Combining (28) and (29) we see that $\operatorname{Re} \phi > 0$. Since $|\phi(z)| = 1$ for $z = e^{ix}$ to show ϕ is continuous it is only necessary to show that it has a continuous argument. The only possible trouble could be at $x = 0$, but it is clear from (28) and (29) that as $x \to 0$ from either direction $\phi(e^{ix}) \to 1$. This shows that ϕ satisfies (26) and completes Example (ii).

Reference

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